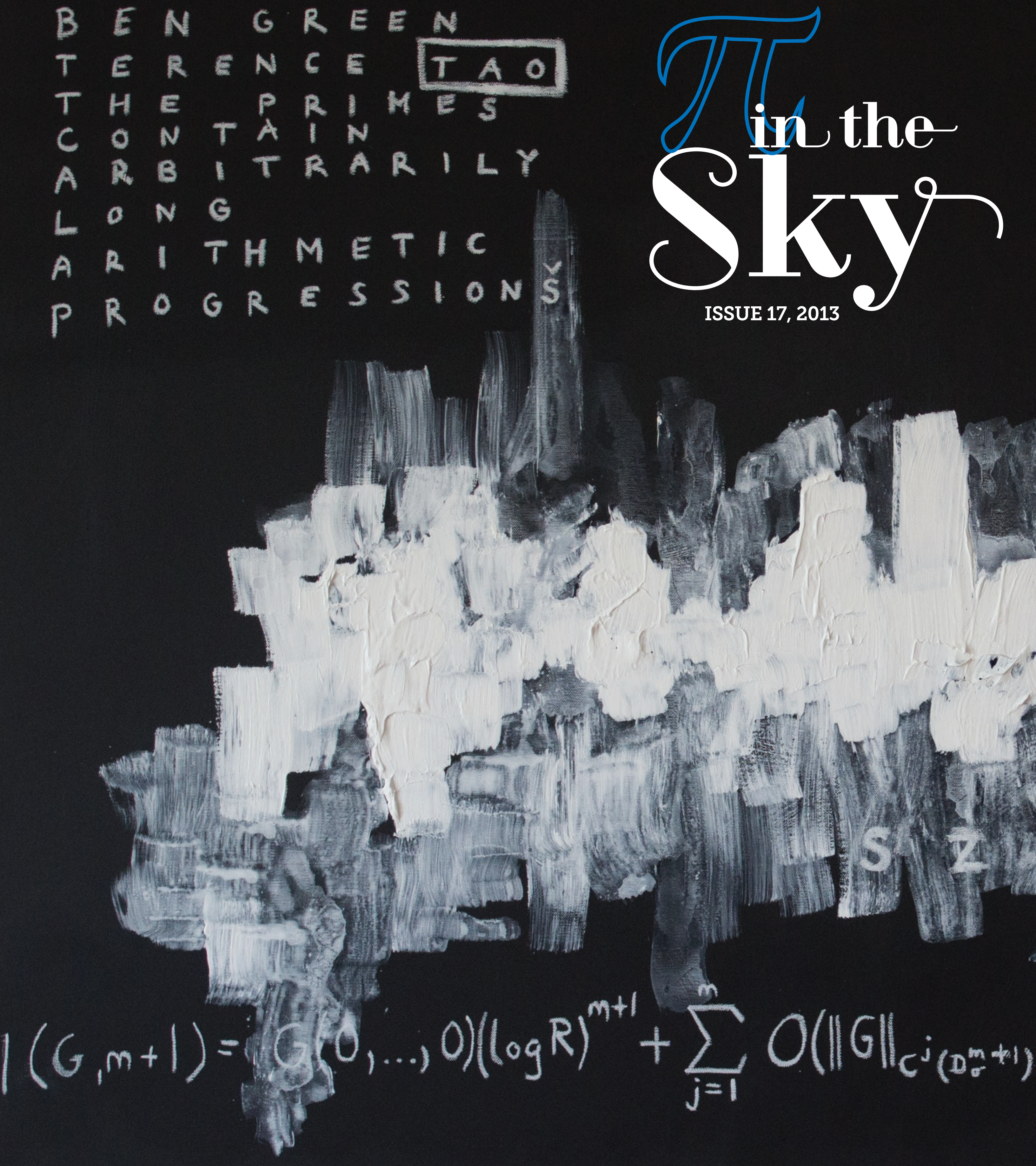
π in the Sky

ISSUE 17, 2013



Pacific Institute *for the* Mathematical Sciences

Pi in the Sky

Issue 17, 2013

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Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

Welcome to Pi in the Sky!

The Pacific Institute for the Mathematical Sciences (PIMS) sponsors and coordinates a wide assortment of educational activities for the K-12 level, as well as for undergraduate and graduate students and members of underrepresented groups. PIMS is dedicated to **increasing public awareness** of the importance of mathematics in the world around us. We want young people to see that mathematics is a subject that opens doors to **more than just careers in science**. Many different and exciting fields in industry are eager to recruit people who are well prepared in this subject.

PIMS believes that **training the next generation of mathematical scientists** and **promoting diversity within mathematics** cannot begin too early. **We believe numeracy is an integral part of development and learning**.

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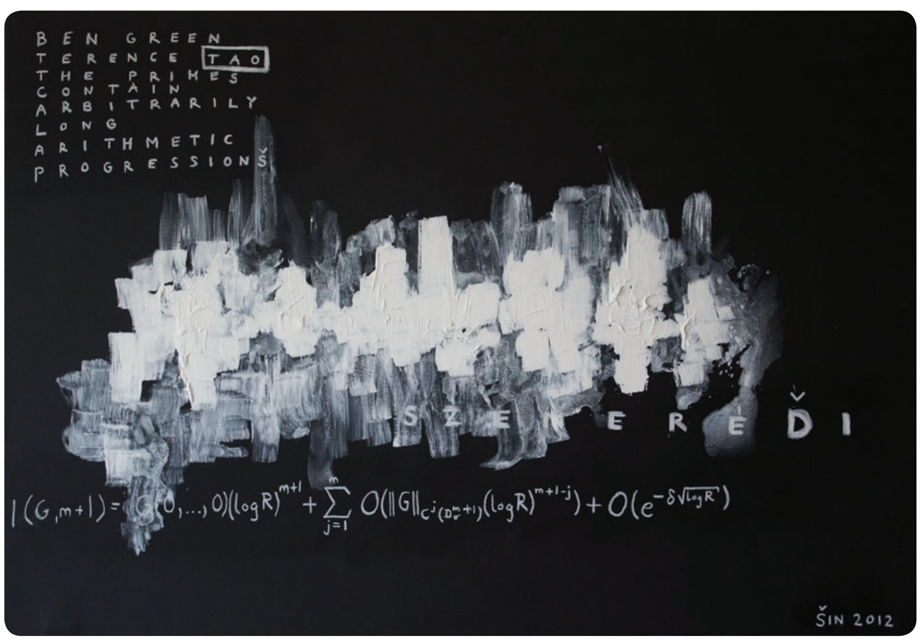
A NOTE ON THE COVER ART



The Green-Tao Theorem with Endre Szemeredi (acrylics and oil pastels on canvas, 110X160cm, 2012, currently in MODESSQE Collection, Poland)

"Portrait of a mathematician as an equation written in chalk on a blackboard? They say that everything we create is a kind of self-portrait, so perhaps a perfect equation is, in fact, the most genuine portrait of its author." Dominik Lejman, about the painting.

Oliver Šin is a Hungarian contemporary painter who combines Neo-expressionism, Street Art, Dadaism, Conceptual and Political Art with science. He has painted scientific formulas on a variety of surfaces including canvas, guitars, girlfriend's shoes and more. When he doesn't paint, he plays the guitar and/or travels his camera in hand.

Editorial
PRIME TIME NEWS
BY ANTHONY QUAS

Prime numbers, those numbers greater than 1, like 2; 3; 5; 7; 11; 13 etc., that have no positive integer divisors other than one and the number itself, have a long history. Indeed they were studied over 2000 years ago by Euclid, and possibly before that. They can be seen as the 'multiplicative building blocks of the integers' because every integer greater than 1 can be written as a product of prime numbers in exactly one way, up to reordering (i.e. we don't distinguish between writing 12 as 2x2x3 and 3x2x2). This fact is sometimes called the 'fundamental theorem of arithmetic' and was proved by Euclid. Euclid also showed that there are infinitely many prime numbers.

You might think that, having been studied for so long, there wouldn't be anything new to ask about prime numbers. In fact this is not the case and some of the best mathematicians in the world are still working on prime numbers.

Here's a well-known question which no one knows how to answer, it is called a *conjecture* because it's widely believed to be true but hasn't been proven. It is also attributed to Euclid.

Conjecture. (Twin Prime Conjecture) *There are infinitely many positive integers n such that both n and n + 2 are prime.*

It's easy to find lots of examples: 3 and 5; 5 and 7; 11 and 13; 17 and 19; 29 and 31 etc. The 100000th pair of twin primes is 18409199 and 18409201. Of course making a long list just shows that there are lots of twin primes, but to prove the conjecture, you'd need to show that the list goes on forever.

A proven fact about prime numbers is the so-called 'Prime Number Theorem.' This asks *how many prime numbers are there up to a number n?* The number of prime numbers up to n is sometimes written $\pi(n)$ and the prime number theorem tells us roughly how big $\pi(n)$ is. The Prime Number Theorem was proved (separately) by Hadamard and de la Vallée Poussin at the end of the 19th Century.

Theorem. (Prime Number Theorem) *The number of primes up to n, $\pi(n)$, is approximately $n/\log n$.*

By approximately, I mean that as n gets larger, the accuracy of the approximation gets better and better (formally the relative error, $(\pi(n) - n/\log n)/\pi(n)$, approaches 0 as n approaches ∞). There are 78498 primes up to 1,000,000, whereas 1000000/log 1000000 is about 72382, a 7.7% error. Up to 1,000,000,000 there's a 5.1% error and up to 10^{12}, there's only a 3.8% error.

A consequence of the theorem is that the further along the integers you look, the fewer prime numbers you find. However, by the old result of Euclid, no matter where you start there are always more to be found.

One remarkable feature of the Prime Number Theorem is the appearance of the logarithm. It's more or less true that logarithms only show up when you're doing calculus. That is, you're working with continuous problems. But we're talking about natural numbers, which are about as far from continuous objects as you can get. *So where does the logarithm come from?*

It's an amazing fact that even though prime numbers are 'discrete' objects (that is they have gaps between them), they can be studied using the techniques of calculus. These ideas go back to the Swiss and German mathematicians Euler (18th Century, pronounced 'Oiler' not 'Yewler'), Dirichlet and Riemann (19th Century), who pioneered the field of what is now called *analytic number theory* (that is the study of discrete objects such as integers using calculus-like techniques).

Another very well-known unsolved question about prime numbers is the following:

Conjecture. (Goldbach's Conjecture) *Any even number greater than 2 can be written as the sum of two prime numbers.*

Again, it's known to be true (using computers) for all even numbers up to 4×10^{17}, but no one knows how to prove it for all n at once.

So far, everything I've mentioned (apart from the computer experiments) was done over a century ago. But real progress is being made on these ancient questions. I will point out three recent developments.

First, a definition: an *arithmetic progression* is a sequence of numbers such as a, $a+d$, $a+2d$, … , $a+kd$, where a is the first term and d is the common difference. This is a $(k + 1)$-term arithmetic progression. For example 59, 83, 107, 131 is a four-term arithmetic progression with a=59 and d=24. Notice that all of the terms in this progression are prime numbers. They are not consecutive primes (61 is prime for example) – but we've found an arithmetic progression, all of whose terms are prime numbers. It has been conjectured since at least the early 20th Century that the primes contain arithmetic progressions of all lengths.

This is now an *ex*-conjecture as it was shown to be true in 2004 by Ben Green (who was a PIMS postdoctoral fellow at the time) and Terence Tao, mathematicians at the University of Oxford and the University of California at Los Angeles respectively. The result was noted by the committee that awarded Tao the 2006 Fields Medal (the mathematical equivalent of the Nobel Prize). The Green-Tao theorem is depicted on the cover of this issue of Pi in the Sky (a work by the contemporary artist Oliver Šin). In fact the scary-looking formula is the last line of Green and Tao's paper.

Even more recently, just this year, there have been a number of remarkable developments in analytic number theory.

Theorem. (Odd Goldbach Conjecture)
Every odd number $n > 5$ can be written as the sum of three prime numbers.

Notice that the Goldbach conjecture above implies this: if you take an odd number greater than 5 and subtract the prime 3, you're left with an even number greater than 2, which is itself the sum of two primes. Putting it all together, the odd number is the sum of three primes. This theorem had previously been established to be true for all 'sufficiently large' numbers n, which means that it was known to be true for numbers greater than some integer. However that integer was 2×10^{1346}, an enormous number. The proof that it holds for all $n > 5$ is due to the Peruvian mathematician, Harald Helfgott and was circulated earlier this year.

As I mentioned above, one of the best known problems in analytic number theory is the twin prime conjecture. This was, until recently, considered to be far out of reach. This year, the Chinese mathematician, Yitang Zhang, announced the following partial result that has been received with great excitement.

Theorem. (Bounded gaps in primes) *If the primes are numbered $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ etc, then there exist infinitely many n's such that gap between the n^{th} and (n+1)st primes, $p_{n+1}-p_n$, is less than 70,000,000.*

As a corollary of this, the following statement can be obtained.

Theorem. (Repeating gap between primes) *There exists a number g less than 70,000,000 such that there are infinitely many primes p where p and p + g are both prime and have no primes in between.*

You can prove this as follows: list all possible gaps 2,4,6,8,…, 6999998 (you only have to list even gaps because since all but one prime number is odd, so all but one gap is even). Then either there are infinitely many gaps between primes of size 2 or not. If yes you're done (and the twin prime conjecture is true). If not look at gaps of size 4. If there are infinitely many of size 4, you're done. If not, try gaps of size 6 etc. By Zhang's theorem, since there are infinitely many gaps of size less than 70,000,000 the number of gaps of size 2 plus the number of gaps of size 4, and so on up to the number of gaps of size 6,999,998 adds up to infinity. So one of these numbers must be infinite!

Starting from Zhang's theorem, Terence Tao (mentioned above) has initiated a *Polymath project*, a project where a large number of mathematicians work collaboratively on a problem to try and reduce the number 70,000,000 as much as possible. As I write this, they have reduced the maximum gap between primes to 4,680, but this is an ongoing project, so watch this space!

The Worst Election Ever: A Look at The Mathematics Of Democracy

BY RACHEL HONG
Rachel Hong wrote this article while she was a high school student at Leland High School, and is currently studying at the University of Pennsylvania.

The debates, platforms and campaigns that come with elections are confusing, but the elections themselves can be almost as hard to understand! Many thanks to Alfonso Gracia-Saz for introducing me to voting theory and to Adam Hesterberg for giving the project idea that led to this article.

The year is 1951. The citizens of Mathematician Nation are electing a new leader and the population's preferences are as follows (Figure 1). These numbers mean that 29 percent of voters have Archimedes as their first choice, Descartes as their second, Cauchy as their third, Euclid as their fourth, Babbage as their fifth and so on.

Let's assume that Mathematician Nation uses the *plurality voting system*. With this system, everyone casts a vote for their first choice and whoever receives the most votes wins. So, Archimedes wins the office.

This seems fair; after all, Canada, the United Kingdom and Mexico, along with many other nations in the world, use this voting system to choose their officials. It turns out, however, that there are a number of unfair problems with using plurality.

1. Why Archimedes shouldn't have won

In this election, what would have happened if Archimedes had been placed in a head-to-head race with each other of the other candidates one by one?

The supporters of Babbage, Descartes, Euclid and Cauchy, who together made up 71 percent of the voters, all preferred Babbage over Archimedes. This group also preferred Cauchy and Euclid over Archimedes. If Archimedes ran against any one of Babbage, Cauchy or Euclid, he would have had only 29 percent of the population's votes and would have lost. If Archimedes ran against Descartes in a head-to-head race, the supporters of Babbage and Cauchy as well as Descartes' supporters would have voted against him and he would have lost, with the support of just 37 percent of the population.

In voting theory, Archimedes is known as the *Condorcet loser*. When put in a two-way election against each of the other four candidates he will always lose. Yet, with the plurality voting system Archimedes would win. So, this system fails the *Condorcet Loser Criterion*, which states that the Condorcet loser must not win. This fairness criterion makes sense, since it seems that while Archimedes had the largest number of people list him as their first choice, many voters greatly disliked him. The part of the population which did not support him (71 percent) ranked him as either fourth or last. If a candidate can win even with 71 percent of the citizens' disapproval there must be a problem.

	1st	2nd	3rd	4th	5th
29%	Archimedes	Descartes	Cauchy	Euclid	Babbage
20%	Babbage	Descartes	Cauchy	Euclid	Archimedes
19%	Descartes	Cauchy	Euclid	Babbage	Archimedes
18%	Euclid	Cauchy	Babbage	Archimedes	Descartes
14%	Cauchy	Euclid	Descartes	Babbage	Archimedes

Fig. 1: Based on a hypothetical distribution created by Alfonso Gracia-Saz.

2. Why Descartes should have won

This was unfair for the voters, but particularly so for Descartes and his supporters. After figuring out that the Condorcet loser criterion was violated in this election, they knew that the *Condorcet winner* criterion—that the Condorcet winner must win for the system to be fair—must have also been broken. If we look closer at the population's preferences, it becomes clear why Descartes' supporters were upset.

Forty-nine percent of the population, the people who support Archimedes and the people who support Babbage, listed Descartes as their second choice. Nineteen percent of voters listed Descartes as their first choice. Clearly, Descartes was quite popular. What would happen if he were placed in a two-way election with each of the other candidates?

Babbage's supporters, Descartes' supporters and Cauchy's supporters, who comprise 53 percent of the population, preferred Descartes over Archimedes. Archimedes' supporters, Descartes' supporters and Cauchy's supporters, who comprise 62 percent of the population, preferred Descartes over Babbage. Archimedes' supporters, Babbage's supporters and Descartes' supporters preferred Descartes to Cauchy and Descartes to Euclid. Against each of the other four candidates, Descartes would have won an absolute majority, making him the Condorcet winner. Because Descartes lost the election despite being the Condorcet winner, the plurality voting system also failed the Condorcet winner criterion.

So far, the 71 percent of voters who did not vote for Archimedes are upset because they ranked him either fourth or last and Descartes' supporters are unhappy because their candidate should have won for being the Condorcet winner. Archimedes is worried about this unrest, but his problems are about to get even worse.

3. The problem with Euclid and why Cauchy should have won

The supporters of Cauchy were suspicious when Euclid decided, at the last minute, to run for office as well. It wasn't that they had a problem with Euclid—in fact, they rather liked him! Cauchy and Euclid had similar beliefs and plans for the future of Mathematician Nation. Unfortunately for their supporters, Cauchy and Euclid were too similar politically and ended up splitting the vote. All voters either liked Cauchy only slightly more than they liked Euclid or liked Euclid slightly more than they liked Cauchy. Cauchy and his supporters are furious about the results. Cauchy had 32 percent of the population's votes before Euclid joined the election and took 18 percent of the voters with him. If Euclid's supporters' votes were combined with his own, Cauchy would have won (Figure 2).

After the election, Euclid admitted that he only ran because he was talked into it by Archimedes' supporters, who knew about another criterion that plurality fails: *Independence of Clones*. Euclid agreed with everything on Cauchy's platform during his campaign and was a better public speaker, dividing the Cauchy voters and preventing Cauchy from winning. In theory, Euclid should not have affected the outcome because he didn't even come close to winning, but because of the flaws in the plurality voting system, he became a relevant alternative and changed the outcome of the election. Plurality is susceptible to strategic nominating, which is unfair.

4. Kenneth Arrow saves the day

By the time Archimedes was about to take his oath of office, everybody but his supporters was protesting. Euclid's supporters were furious because they had been tricked into supporting an irrelevant alternative. Cauchy's supporters felt cheated because they could have won if the votes weren't split.

	1st	2nd	3rd	4th
29%	Archimedes	Descartes	Cauchy	Babbage
20%	Babbage	Descartes	Cauchy	Archimedes
19%	Descartes	Cauchy	Babbage	Archimedes
32%	Cauchy	Babbage	Descartes	Archimedes

Fig. 2: Without Euclid in the running, Cauchy would have won.

Descartes' supporters were angry because their candidate was the Condorcet winner and therefore should have won. Babbage's supporters were annoyed because they greatly disliked Archimedes and knew that the majority of Mathematician Nation's citizens shared this dislike.

The nation was full of unrest and Archimedes feared that he would be pushed out of office. In a panic, Archimedes called Kenneth Arrow, who would later win a Nobel Prize in Economics, a man respected by everyone in Mathematician Nation.

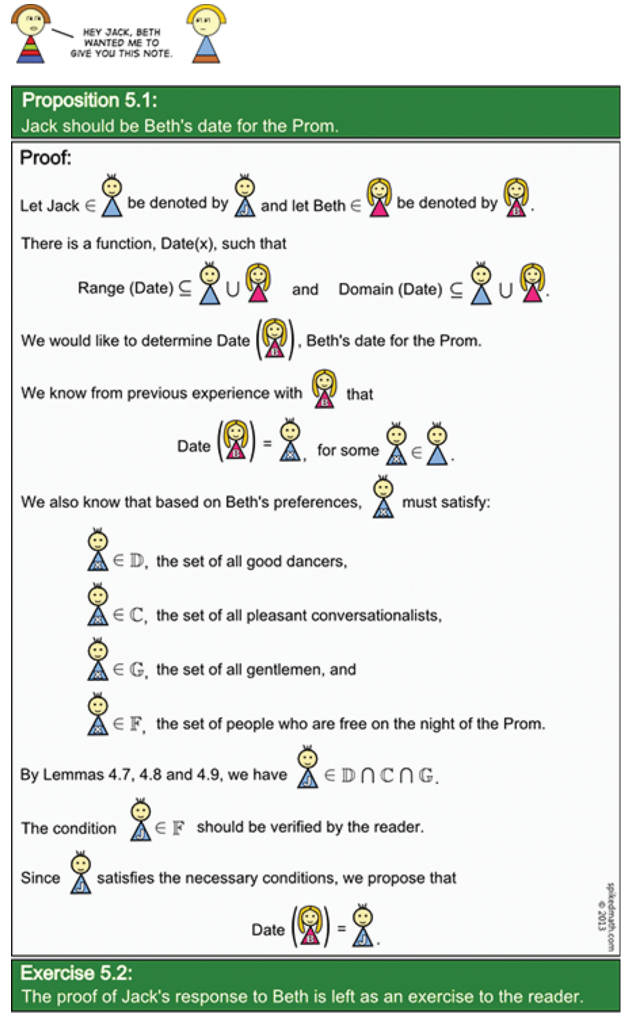
Arrow sent him a book called *Social Choice and Individual Values*, which includes a theorem he first introduced in his PhD thesis. He reminded the citizens of Mathematician Nation that the beauty of math is that it can be used everywhere. Math has applications in technology, its patterns are in nature and the physical sciences require it for every formula. In fact, mathematical logic can even show that all elections are unfair!

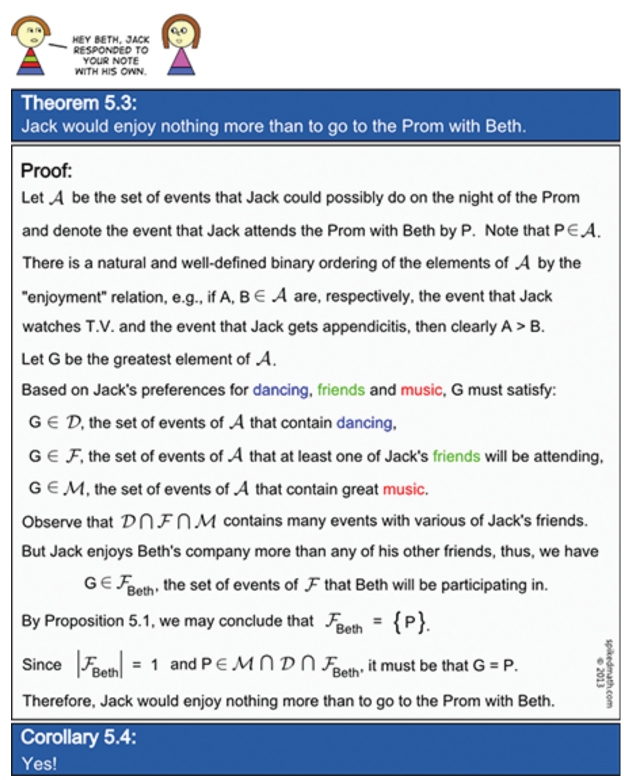
Wait… what?

5. Why nobody should have won

Alas, it is true. Elections are unfair and Kenneth Arrow has proven it. A mathematically derived theorem, Arrow's impossibility theorem, states that no election with more than two candidates can ever be completely fair. In fundamental terms, there is no voting system that can satisfy these fairness criteria: unanimity (if the entire group prefers one candidate to another, the ranking should reflect this), independence of irrelevant alternatives (if a candidate who loses is removed from the election, the outcome should not change) and non-dictatorship (because democracy says so).

Mathematician Nation was saddened by this revelation, but could not deny it. Several mathematical proofs show the truth in Arrow's theorem and the people accepted that no election can ever be completely fair. Luckily, for the next election they avoided such a dilemma. One man became so popular that no other potential candidate dared run against him. Kenneth Arrow for dictator!





Math: The Age-Old Question

BY KENTON KAUPP

Have you ever wondered where math came from? Or who invented numbers? There isn't one answer because math arose independently in many cultures, but we can look at how it developed in one culture and explore what it was like to be a student in ancient times.

One of the most studied areas of the ancient world is Mesopotamia, making it a (relatively) easy case study. Mesopotamia is in the area surrounding the Tigris and Euphrates rivers in modern day Iraq and is part of the Fertile Crescent, an area where grain crops were reasonably easy to grow. Civilization is thought to have begun approximately 12,000 years ago when people transitioned to farming from a hunter-gatherer lifestyle. This area is sometimes referred to as the *cradle of civilization* because it was one of the first places people settled and farmed.

Mesopotamia was an area poor in many resources; there was no metal, stone, hard timber or minerals. Because these materials were only available through trade or conquest, mud and reeds tended to be the materials used in everyday life. Houses, canals and even tools were made using only mud, mud bricks, fired clay and reeds.

As the system of agriculture grew more complex, people began to use tokens for trading to keep track of labour and commodities. The tokens can be thought of as the beginnings of mathematics as they showed both counting and the process of abstraction.

From Tokens to Tablets

Tokens originated 9,000 years ago as simple geometric figures made out of clay with different shapes representing different goods. For example, a cone could represent an amount of grain, while a sphere might represent a different amount of grain and a tetrahedron, an amount of labour. The major advantage of tokens was that they could be counted, manipulated and traded without having to physically move heavy baskets of grain or ornery animals. As time progressed the number of types of tokens, as well as their complexity, increased until there were over 350 types of tokens in circulation, often meaning different things in different areas.

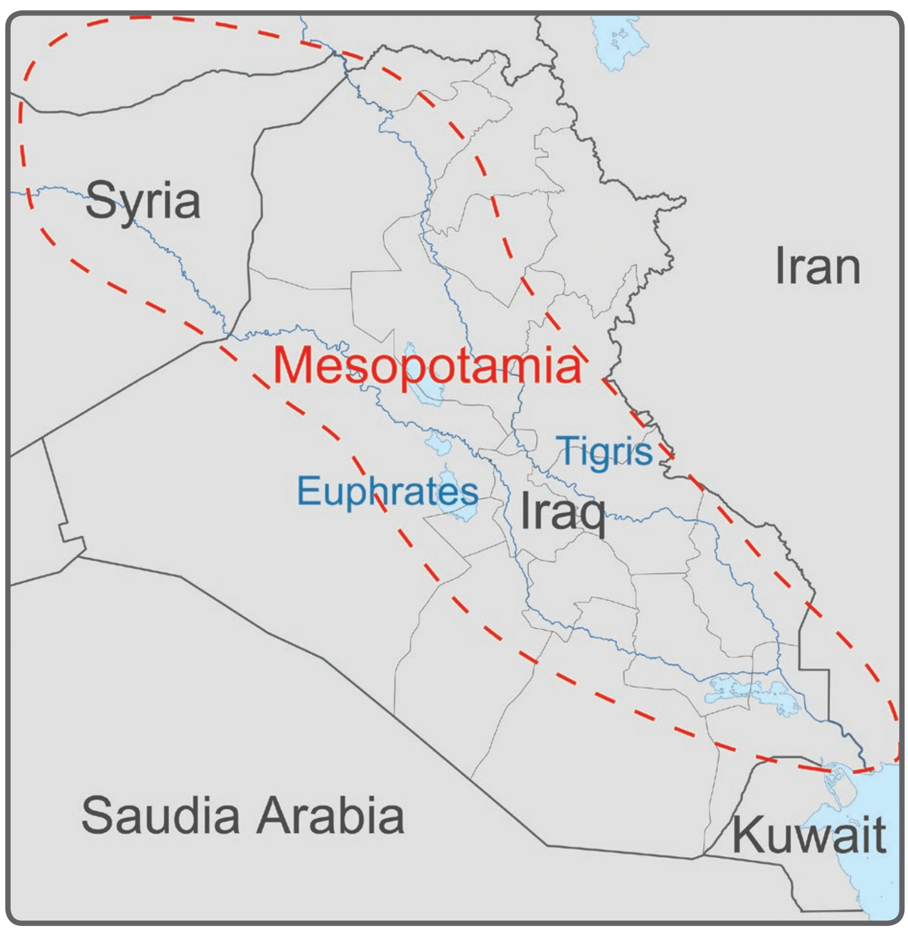
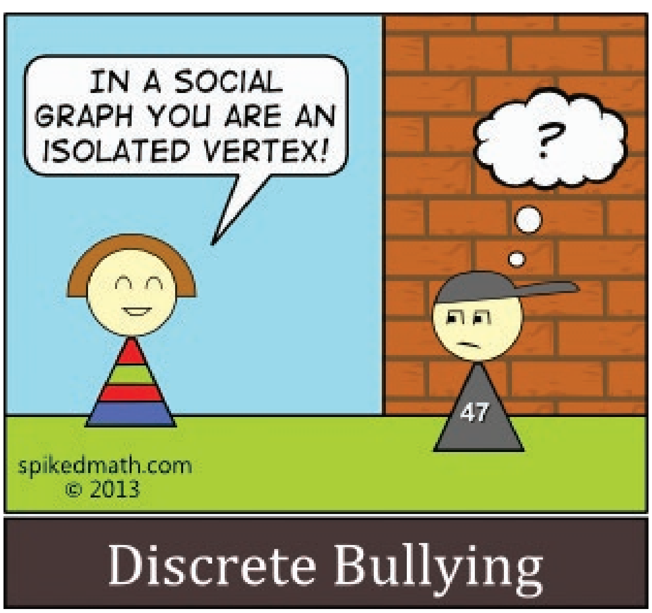


Fig. 1: Mesopotamia was located in modern day Iraq. *Modified and used under GNU Free Documentation License.*

Around 5,300 years ago it became common to deal with debts by sealing tokens in a clay sphere that was marked with a seal which would break if tampered with. This way both parties knew the contents remained the same. Unfortunately, it was difficult to determine what was in the sphere without opening it, so generally an impression of the tokens inside the sphere was made on the outside of the sphere to indicate its contents. Eventually, it was realized that this process had redundant information and so was simplified just using the impressions as the record. Using this system, five grain impressions would be used to represent five units of grain and two oil impressions would be used to represent two units of oil. These impressions on clay are considered the first type of writing in Mesopotamia.

Urbanization began significantly in Mesopotamia around 5,200 years ago as city states began to form. With this development came an unprecedented volume of goods to manage and the writing system needed to evolve to handle this change. First, the impressions were replaced with a sketch made with a pointed stylus allowing a scribe to work more quickly. Next, the one symbol per item strategy gave way to a numeral followed by the type of good, similar to the way we might say "nine sheep" today. This method was advantageous because scribes could represent very large numbers using relatively little writing space.

This number system varied from region to region and sometimes things we would consider unusual would appear.



For example, the set of numerals to count discrete objects like animals was different from one used for continuous measurements such as areas and volumes. Many of the measurement units were not standardized and there were different sized conversion factors between units which is similar to the conversion factors seen in the US today.

LENGTH	VOLUME
1 FT = 12 IN	1 CP = 8 OZ
1 YD = 3 FT	1 PT = 2 CP
1 MI = 1760 YD	1 QT = 2 PT

LENGTH	VOLUME
1 CUBIT = 3 DOUBLE-HANDS	1 BAN = 10 SILA
1 REED = 6 CUBITS	1 BARIGA = 6 BAN
1 ROD = 2 REED	1 LIDGA = 4 BARIGA

Fig. 2: The US measurement system has various conversion factors. Adding to this complexity, the volume units have a different meaning in the US and Canada. Ancient measuring systems would have seen much of this type of variation. The second table shows a measurement system used just over 5,000 years ago. A cubit is about 50 cm and a sila is about 1 L.

Four thousand years ago, Mesopotamia's government grew very large and centralized which resulted in the numeric system being standardized. This included writing in cuneiform text and using a sexagesimal[1] place value system. The period following this change is often called the Old Babylonian Period which lasted from about 4,000 to 3,600 years ago and is the period for which we have the most information. This is largely because of the endurance of the clay tablets that were commonly used for writing, which withstand weathering better than other mediums such as wax or papyrus.

Like a Mesopotamian Student

An Old Babylonian school typically consisted of a few rooms and a courtyard. These looked like the surrounding houses, except they contained many more tablets than a house would.

1. Sexagesimal refers to base 60. Remnants of this system can still be seen in how we measure angles and time. 6 x 60 = 360°, 60 seconds make a minute and 60 minutes make an hour.

Using clay as a writing medium had a few challenges. The clay needed to stay wet to be written on so a basin of water would have been nearby.

A student needed to write fast enough to complete their work before the clay dried and couldn't fix mistakes afterwards. Clay tablets were bulky, so storage was likely an issue.

Discarded tablets were often used as building filler. To reduce the number of tablets, only the important information was kept while the 'work' was done on a separate hand tablet which was reusable.

Most students were training to become scribes or bureaucrats. It appears that these students spent a large amount of time copying tables from an instructor. These tables could include things like names of animals or places, translations between languages and mathematical tables. This served a dual purpose; the student became more familiar with the material, while creating a personal library for later use. Much of the mathematics taught used multiplication, reciprocal, square, square root and conversion tables to simplify procedures. Usually mathematics was taught in a way that each problem used a specific algorithm to solve it. So for problem type A, use solution type A. Evidence for this is apparent because students would often show unnecessary steps such as multiplying by 1.

Take a step back in time and try some mathematics Babylonian style.

Cuneiform Numbers

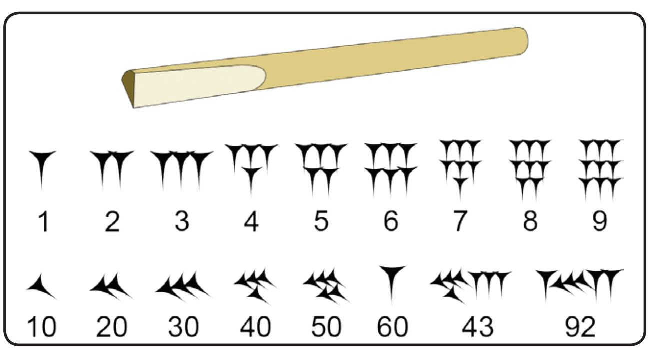


Fig. 3: A wedge shaped stylus was used to write in cuneiform. The cuneiform number system used during the Old Babylonian Period only required two unique symbols. The style of writing varies somewhat from person to person and over time just as is true today. Notice how the numbers progress and that 1 and 60 have the same symbol.

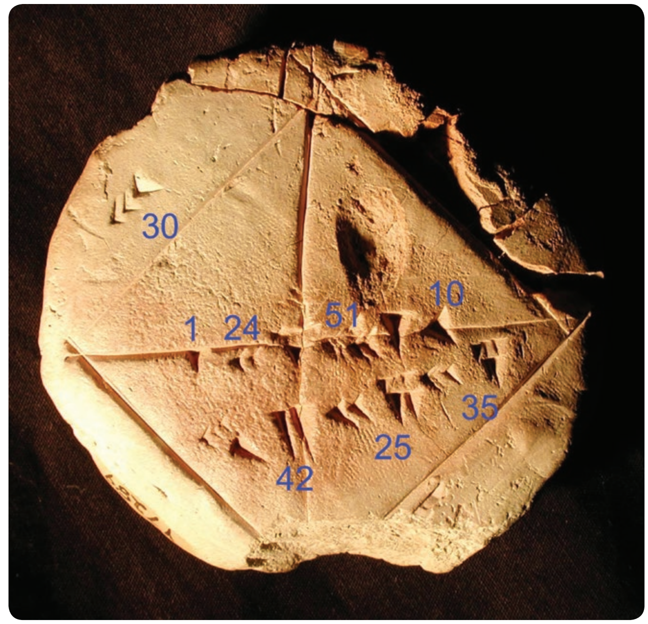


Fig. 5: Here is a tablet with cuneiform numbers c. 3,800 – 3,600 years ago. Which numbers can you identify correctly? The tablet is part of the Yale Babylonian Collection. Photograph courtesy of Bill Casselman, http://www.math.ubc.ca/~cass/Euclid/ybc/ybc.html.

During the Old Babylonian Period, numbers were represented using two symbols: one to represent 1 and one to represent 10 as shown in Figure 3. To represent 2, you used two 1s and to represent 34 you used three 10s and four 1s. Once you reached 60, you added 1 to the `place' to the left, just like after 9, you add a 1 to the left to get 10. To represent sexagesimal numbers in this article, spaces are used to represent place value and a semicolon will be used to separate whole numbers from fractional ones[2]. For example, the sexagesimal value 12 43 6 represents

$$12 \times 60^2 + 43 \times 60 + 6 = 45{,}786$$

in decimal notation and 12; 43 6 represents

$$12 + 43 \times 60^{-1} + 6 \times 60^{-2} \approx 12.718.$$

Using a place value system allows you to easily represent large numbers. This is an advantage over a system like the Roman numerals which requires different symbols as numbers get larger. Another advantage of using base 60 is that 60 has 10 proper divisors greater than 1 (2, 3, 4, 5, 6, 10, 12, 15, 20, 30) whereas 10 only has two (2, 5).

2. There were notations for simple fractions such as 1/2 and 1/3, however, for this article all numbers will be represented using numbers, spaces and semicolons. 1/2 will be written ; 30, 1/3 will be written ; 20 and so on.

This makes division by these numbers easier just as dividing numbers by 2 or 5 is much easier than other numbers when using the decimal system.

SEXAGESIMAL	1 0 0	4 37 46 40	8 51; 15
DECIMAL	216 000	1 000 000	531.25

Fig. 4: Some equivalent numbers written in both sexagesimal and decimal.

Two things about the Old Babylonian system can lead to ambiguity. The first is that the Mesopotamians did not have a symbol for 0 to act as a place holder. This means that 1 can represent (using zeros) 1, 1 0 or 1 0 0 as well as other numbers and 12 21 can represent 12 21, 12 0 21 or 12 0 0 21 0 as well as other numbers. A second problem is that there was no notation for a decimal and so 34 5 39 can represent many different values such as 34; 5 39, 34 5; 39, 34 5 39. Eventually, notations were produced to solve both of these problems; a space or spaces were used to represent zeros and a symbol, 𒑱, was used to indicate place value.

ADDITION AND SUBTRACTION

Addition and subtraction were two mathematical procedures that did not rely on tables. Typically, no intermediate steps were shown for these types of problems, however, algorithms similar to those used today, that included carrying and borrowing were probably used. Figure 6 shows an example of addition and subtraction using sexagesimal numbers.

	1		
	58	19	31
		32	18
1			
2	3	47	
3	2	38	49

17	30+60	+60
~~18~~	~~31~~	
	51	; 19
17	39	; 41

Fig. 6: The addition 58 19 31 + 32 18 + 2 3 47 0 = 3 2 38 49 and the subtraction 18 31 – 51; 19 = 17 39; 41. Notice that `carrying' is used when a column sums to more than 60 and you `borrow' 60 at a time.

MULTIPLICATION

Most students would have had basic multiplication tables memorized up to 12 × 12, just like you (hopefully). Multiplication beyond this would have relied on a table. Using the multiplication table in Figure 7 to multiply 30 × 12 48 could have been done in parts by looking up the values separately and summing the results.

$$30 \times 12\ 0 = 6\ 0\ 0$$
$$30 \times 40 = 20\ 0$$
$$30 \times 8 = 4\ 0$$
$$6\ 0\ 0 + 20\ 0 + 4\ 0 = 6\ 24\ 0$$

Note that place value needs to be accounted for when adding the products.

1	30
2	1
3	1 30
4	2
5	2 30
6	3
7	3 30
8	4
9	4 30
10	5
11	5 30
12	6
13	6 30
14	7
15	7 30
16	8
17	8 30
18	9
19	9 30
20	10
30	15
40	20
50	25

60	
2	30
3	20
4	15
5	12
6	10
7; 30	8
8	7; 30
9	6; 40
10	6
12	5
15	4
16	3; 45
18	3; 20
20	3
24	2; 30
25	2; 24
27	2; 13 20
30	2
32	1; 52 30
36	1; 40
40	1; 30
45	1; 20
48	1; 15
50	1; 12
54	1; 6 40

Fig. 7: Here is a multiplication table (left) and a reciprocal table (right) that a student would have been familiar with. Note that the reciprocal table only includes values that can be written using a terminating sequence. For example, 60 ÷ 7 = 8.5714… and so it is not included. 7 30 was included on most reciprocal tables, likely because it was a commonly used fraction.

Division

Multiplying by a reciprocal was used for division. Reciprocal tables can be hard to remember, so it was common for a student to use a reciprocal table like that shown in Figure 7. In base 60, just like in base 10, you can end up with reciprocals that have infinitely long decimal representations. The only sexagesimal numbers with finite reciprocals have factors that are powers of 2, 3 and 5. These are now called *regular numbers*. It seems that the ancient math instructors went out of their way to make sure solutions required only regular numbers. As with the other operations, place value needed to be accounted for when completing a division. A sample is shown.

$$
\begin{aligned}
37\ 36 \div 24 &= 37\ 36 \times ;2\ 30 \\
&= 37\ 36 \times ;2 + 37\ 36 \times ;0\ 30 \\
&= 1\ 15;12 + 18;48 \\
&= 1\ 34
\end{aligned}
$$

The Quadratic Equation

Geometric problems were common for Old Babylonian students. The following problem is adapted from a translation given in A. E. Berriman's article *The Babylonian Quadratic Equation* (1956). The problem and its solution in modern notation are similar in style to what an Old Babylonian student would have experienced.

Notice that there is no explicit question in the problem, it is implied that the student is supposed to determine the *square line* which is referring to the side length. The solution is given to the student and is meant to be a model for this type of problem. Generalized solutions were not used.

- The surface and the square line I have accumulated: 12.

$$
\begin{aligned}
&1 \\
;30 \times 1 &= ;30 \\
;30^2 &= ;15 \\
;15 + 12 &= 12;15 \\
&= 3;30^2 \\
3;30 - ;30 &= 3
\end{aligned}
$$

You may have found that the problem above can be written as $x^2 + x = 12$ where the student is meant to determine the value of x. One interpretation of this solution is found in Figure 8 and below.

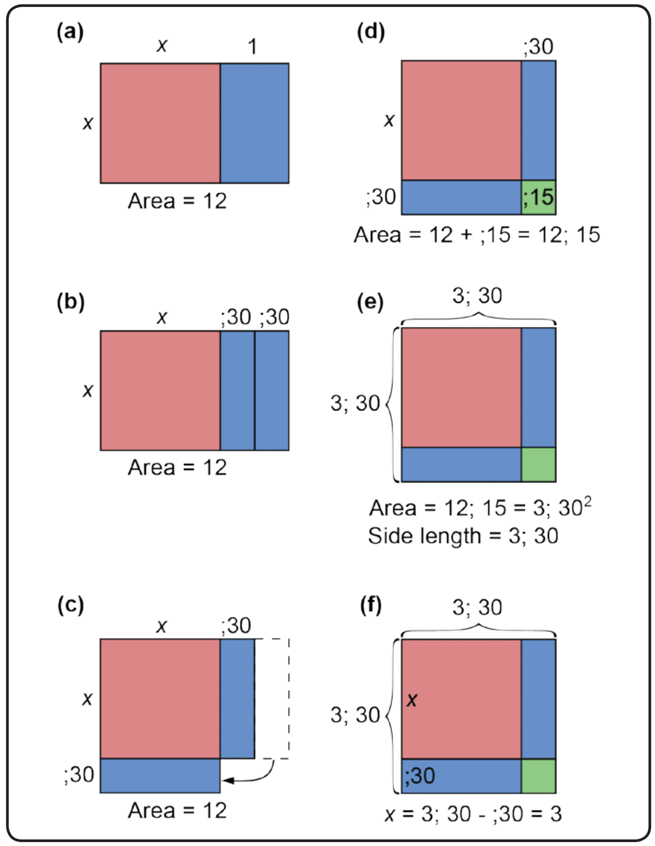


Fig. 8: This may have been how a student interpreted the solution to a quadratic equation.

a. Draw a rectangle with dimensions x, x+1. Separate it into a square with side length x (red) and a rectangle (blue).

b. Divide the rectangle in half to get two blue rectangles with widths of ; 30.

c. Move one of the blue rectangles to the bottom of the square.

d. Determine the area of the missing square (green), ;15 in this case and the area of the large square, 12; 15.

e. Use the area of the large square to determine the side length of the large square, 3; 30 in this case.

f. Subtract the width of the blue rectangle from the length of the large square to determine the unknown length, $x = 3$.

Try to correlate the steps in this solution to those in the Old Babylonian solution.

Square Roots

Look back to Figure 5. This tablet is thought to be the work of a student because the writing and the tablet are fairly large. What do you suppose the significance of the tablet was? As a clue, convert to decimals: ; 30 = 0.5, 1; 24 51 10 ≈ 1.4142 and ; 42 25 35 ≈ 0:7071. It looks like the student is using an approximation of $\sqrt{2}$ to determine the length of the diagonal of a square with side ; 30. There is no work shown by the student, but he likely looked up the value of $\sqrt{2}$ from a table. What step(s) could he have used to determine the diagonal length?

It is clear from the tablet shown in Figure 5 that Old Babylonians were familiar with square roots, but how did they calculate them? No outright calculations have been found but some clues suggest the method shown in Figure 9 and below.

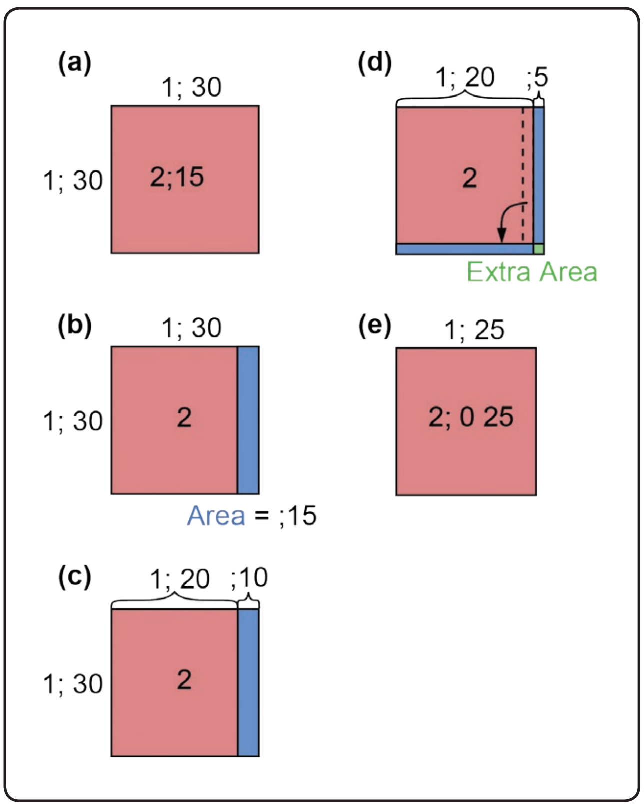


Fig. 9: Here's one iteration of how $\sqrt{2}$ may have been approximated. The diagram is not to scale.

a. Begin with an estimate. 1; 30 is a reasonable start as 1; 30^2 = 2; 15 which is fairly close to 2. Draw a square with side length 1; 30.

b. This is an overestimate so you can separate this square into rectangles of area 2 and ;15.

c. Determine the unknown length of the small rectangle by dividing the area by the known length, ; 15 ÷ 1; 30 = ; 15×; 40 = ; 10.

d. Next, divide this rectangle in two and use one to cover the bottom of the original estimate square.

e. This gives you an overestimate of 1; 25, 1; 25^2 = 2; 0 25. Use this as your new estimate and repeat the process as until you reach a desired level of accuracy.

A similar process can be used for underestimates by adding a rectangle to the underestimate, so they sum to 2.

******

Giving word problems to students was a favourite activity of Old Babylonian teachers. Normally, students would be given some information and expected to compute an unknown value. Just as it is true today, many of the word problems had no real-world usefulness. Interested readers are encouraged to look into some of these in Elenor Robson's *Mesopotamian Mathematics* (2007), which includes translations for word problems as well as many other mathematical texts.

In many ways, the math learned by the ancient Mesopotamians was similar to the math students learn today. Students were given examples to work from and problems to solve, the work was more important than the answer and unrealistic problems were used to teach thinking skills. It is amazing to think that in some respects, a student's experience four millennia ago was similar to a student's experience today.

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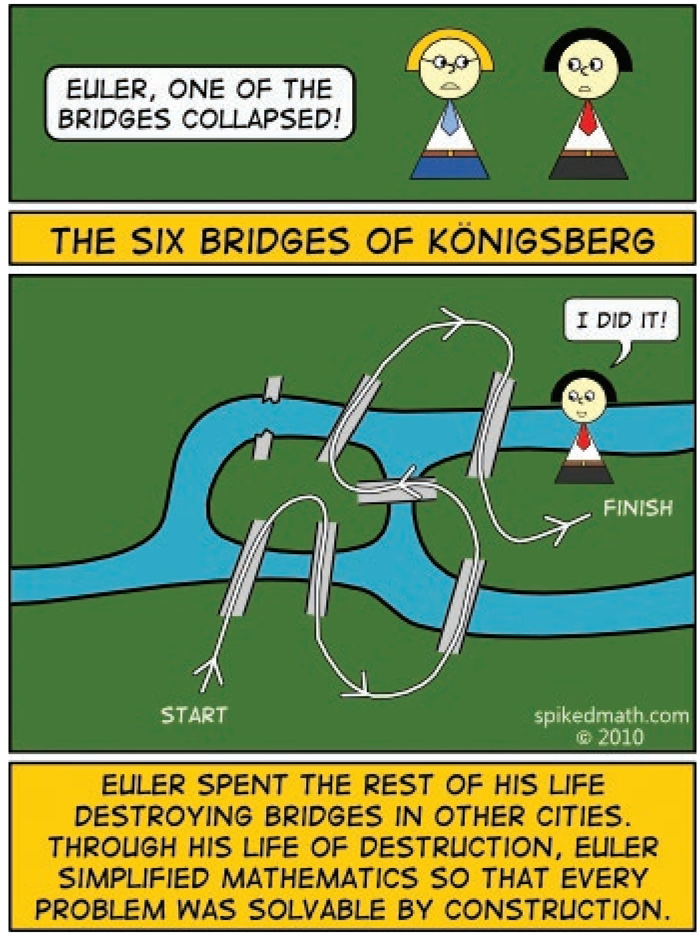
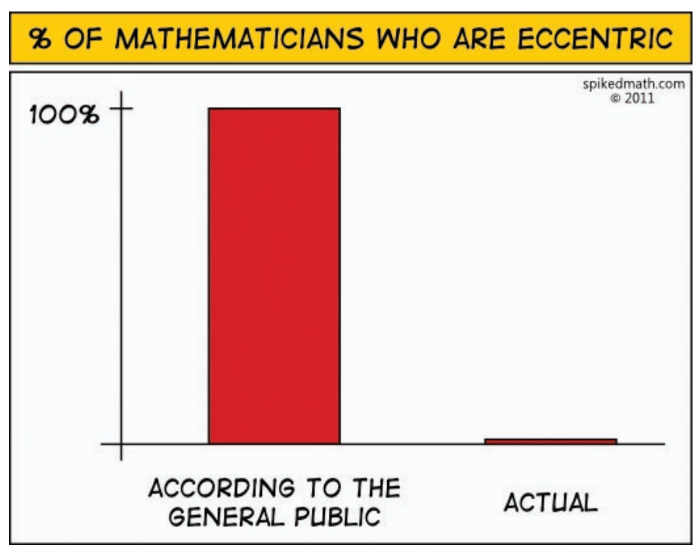




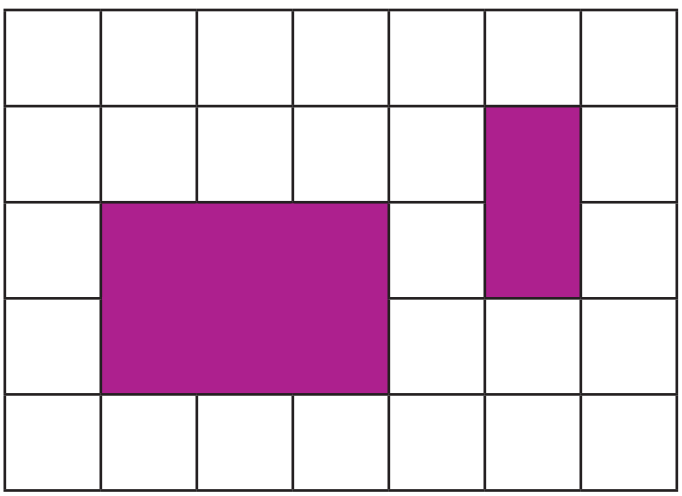
THINK OUTSIDE THE ~~BOX~~ GRID

BY CHRISTINE LI, GUNN HIGH SCHOOL, CALIFORNIA

Christine Li wrote this article while she was a high school student at Gunn High School, California and is currently studying at Princeton.

When given an x by y grid with two rectangular black holes that do not touch one another, how do you figure out the number of unique paths from the bottom left vertex to the top right vertex of the grid? A path is valid only if every step of the path goes either towards the right or towards the top and not through any black holes. A step in the path, however, may go along the edges of the black holes. Let us start with an example problem.

B



A

Fig. 1

GIVEN A 7X5 GRID WITH TWO HOLES, HOW MANY DIFFERENT PATHS ARE THERE FROM A TO B? (FIG. 1)

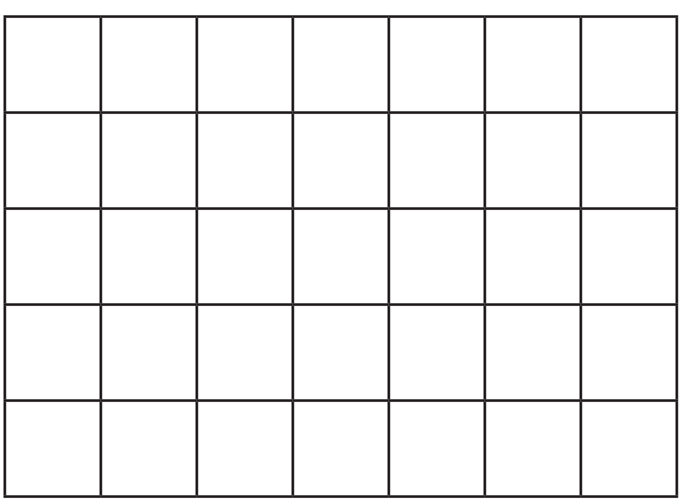
At first glance, the problem may stump you. Should the grid be separated into sections with black holes and without black holes? Should complimentary counting be used: (total number of paths from A to B ignoring the black holes) – (number of paths from A to B that go through either/both of the holes)? At this point, there seems to be no obvious way to go about solving the problem.

Before we dive into the problem, let us match the grid with a Cartesian plane, placing the origin at the bottom left vertex of the grid, the x-axis along the bottom edge of the grid and the y-axis along the left edge of the grid. Then, the coordinates of point A are (0,0) and the coordinates of point B are (7,5).

SIMPLIFY

Let us simplify the problem by removing the black holes from the grid (Fig. 2). There are two methods to find the number of unique paths from A to B.

B



A

Fig. 2

METHOD 1

Let each coordinate point have a property k where k is the number of possible paths from (0,0) to that coordinate point, and let $k@(a,b)$ denote the k value of the point with coordinate (a,b).

INITIAL CONDITIONS

$k@(0,0)=1$, because there is only 1 way to get from (0,0) to (0,0). Also for any x, $k@(x,0)=1$ and for any y, $k@(0,y)=1$ because there is only 1 way to get from (0,0) to any point along the left or bottom edges of the grid.

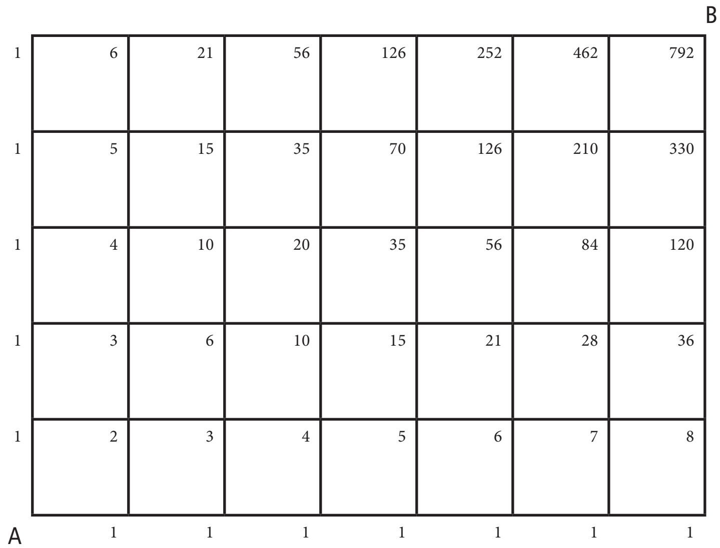


Fig. 3

As stated above, $k@(1,0)=1$ and $k@(0,1)=1$. Then, $k@(1,1)=k@(1,0)+k@(0,1)$ because each path from (0,0) to (1,1) must go through either (1,0) or (0,1).

We can generalize this: For any **(x,y), $k@(x,y)=k@(x\text{-}1,y)+k@(x,y\text{-}1)$** [Eq. 1] because each path to (x,y) must go through either $(x\text{-}1,y)$ or $(x,y\text{-}1)$.

This is called a recurrence relation. (A good introduction to this topic is given in Chapter 7 of Richard A. Brualdi, Introductory Combinatorics, 5th edition [ISBN-10: 0136020402].)

Using Eq. 1, we can find the k value at every coordinate point in the grid. We see that $k@(7,5)$ is 792. (Fig. 3)

METHOD 2

Notice that any path from A to B goes through 12 steps, which consist of 5 steps upwards and 7 steps to the right. Let 'U' denote a step upwards, and 'R' denote a step to the right. We can look at each path from A to B as a permutation of 5 'U's and 7 'R's. An example of a valid path from A to B would be 'URRURURRRUUR'.

The number of different paths from A to B is then $\binom{12}{5} * \binom{7}{7} = \mathbf{792}$, because there are $\binom{12}{5}$ ways to choose which 5 places in the permutation to place the 'U's, and after the 'U's are placed, there are $\binom{7}{7}$ ways to choose the remaining 7 places in the permutation to place the 7 'R's.

Another way to look at the number of different paths from A to B is $\binom{12}{7} * \binom{5}{5} = \mathbf{792}$, because there are $\binom{12}{7}$ ways to choose which 7 places in the permutation to place the 'R's, and after the 'R's are placed, there are $\binom{5}{5}$ ways to choose the remaining 5 places in the permutation to place the 5 'U's.

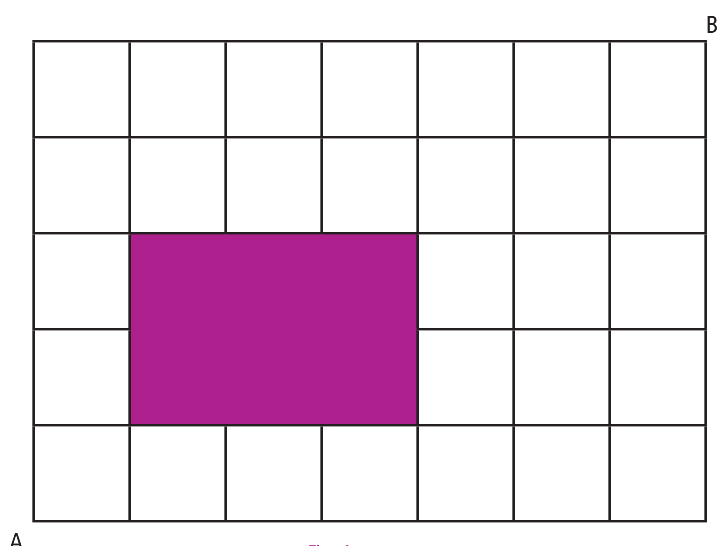


Fig. 4

ADD A COMPLICATION

Now, let us add one hole to the problem. Assume there is a black 3x2 hole in the middle of the grid as shown. How many different paths are there from A to B? (Fig. 4)

METHOD 1

As stated before, we can use the following equation to get the number of paths from point (0, 0) to point (x, y) in a grid without holes.

$k@(x,y)=k@(x\text{-}1,y)+k@(x,y\text{-}1)$

Now, can we apply this equation to a grid with holes? The answer is yes, but with a few added ***black hole rules***:

> **BLACK HOLE RULES**
> 1. k@(x,y)=k@(x-1,y) if the path from (x,y-1) to (x,y) falls inside the hole.
> 2. k@(x,y)=k@(x,y-1) if the path from (x-1,y) to (x,y) falls inside the hole.
> 3.k@(x,y)=0 if both paths fall inside the hole (This rule applies to all points inside the hole).

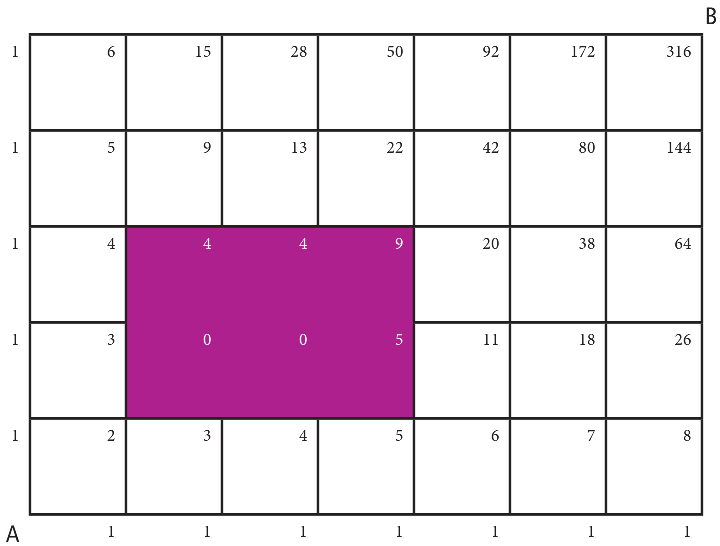


Fig. 5

RULE 1: the only way to get from (0,0) to (x,y) is through $(x\text{-}1,y)$, as in the case of point at (2,3): k@(2,3)=k@(1,3)=4.

RULE 2: the only way to get from (0,0) to (x,y) is through $(x,y\text{-}1)$, as in the case of point (4,2): k@(4,2)=k@(4,1)=5.

RULE 3: there are 0 paths from (0,0) to (x,y), as in the case of (2,2).

By this logic, we can fill in the rest of the grid from k@(0,0) to k@(7,5), and we see that k@(7,5)=316. (Fig. 5) Therefore, there are **316 paths** from A to B.

COORDINATES	# OF PATHS FROM (0,0) TO POINT	# OF PATHS FROM POINT TO (7,5)	# OF PATHS FROM (0,0) TO (7,5) THROUGH POINT
(0,4)	$\binom{4}{4}=1$	$\binom{8}{1}=8$	1*8=8
(1,3)	$\binom{4}{3}=4$	$\binom{8}{2}=28$	4*28=112
(4,1)	$\binom{5}{1}=5$	$\binom{7}{4}=35$	5*35=175
(5,0)	$\binom{5}{0}=1$	$\binom{7}{5}=21$	1*21=21

METHOD 2

We can see that any path from A to B that does not pass through the hole must go through one of the following points: (0,4), (1,3), (4,1), (5,0)—call them 'crucial points'. But how do we determine which points are considered 'crucial points'?

Suppose a rectangular black hole's top left vertex is at (x_m,y_m) and its bottom right vertex is at (x_n,y_n). Draw a diagonal line in the northwest direction starting from point (x_m,y_m) until it either hits another black hole or the corner or edge of the grid. Then draw a diagonal line in the southeast direction starting from point (x_n,y_n) until it either hits the vertex or edge of another black hole or the vertex or edge of the grid. All points on these two lines constitute the set of crucial points for this black hole.

Each path is essentially broken into two parts: from (0,0) to one of the crucial points and from that crucial point to (7,5). To get the total number of paths from (0,0) to (7,5), paths that pass through each of the crucial points need to be considered.

In order to get the number of paths that pass through each crucial point, we need to multiply the number of paths from (0,0) to the crucial point by the number of paths from that point to (7,5) because each full path is a combination of the two sub-paths and must go through both paths (see table above).

This uses the multiplication principle[1]. Then, the number of paths from A to B that pass through each of the crucial points is added together to get the final answer because the paths through different crucial points are mutually exclusive. *This uses the addition principle*[1]. Thus, we get a total of 8+112+175+21=**316 paths.**

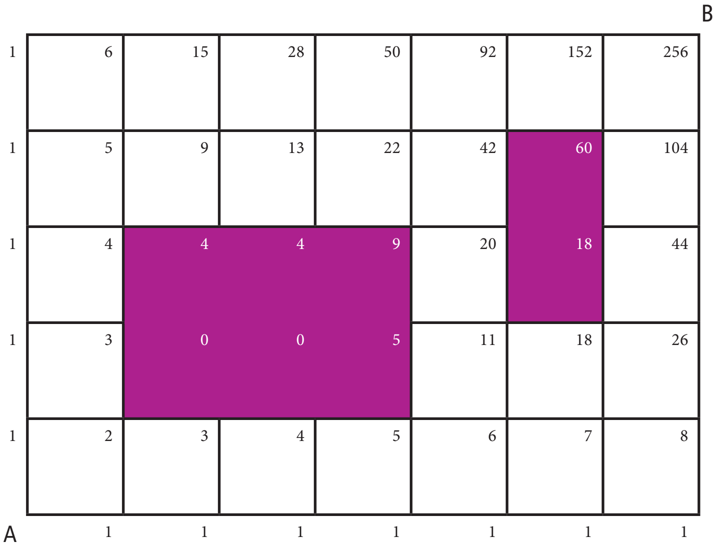


Fig. 6

ADD ANOTHER COMPLICATION

Now, we add the second hole and are back to our original problem (Fig. 1). How many unique paths are there from A to B?

The same two methods used for one hole can also be applied to the problem with two holes.

METHOD 1

Find the k value for each point in the grid by applying the equation and ***black hole rules***. The number of paths from A to B is **256 paths** (Fig. 6).

METHOD 2

The crucial points associated with the first hole stay the same: (0,4), (1,3), (4,1), (5,0). But now that there is a second hole, there are crucial points associated with this hole as well: (4,5), (5,4), (6,2), (7,1).

Now let us organize these crucial points into two categories: 'Category 1' and 'Category 2'.

CATEGORY 1: If all possible paths were to be drawn from (0,0) to this point, the paths would not pass through any other crucial point.

CATEGORY 2: If all possible paths were to be drawn from (0,0) to this point, at least one of the paths would pass through a crucial point in Category 1.

To generalize for any $x>1$:

CATEGORY X: If all possible paths were to be drawn from (0,0) to this point, at least one of the paths would pass through a crucial point in Category (x-1).

So, each path goes through exactly one crucial point in Category (x-1).

Applying this to our problem, we get:

CRUCIAL POINTS 1: (0,4), (1,3), (4,1), (5,0).

CRUCIAL POINTS 2: (4,5), (5,4), (6,2), (7,1).

All paths from (0,0) to (7,5) must go from (0,0) → crucial point 1 → crucial point 2 → (7,5).

To calculate the total number of paths from (0,0) to (7,5), all crucial points need to be considered. Using the same method shown in the simpler version of the problem with only one hole, we fill in the table as below.

1. R. A. BRUALDI, Introductory Combinatorics (5th Edition) *Pearson* (2008)

Abbreviations: cp1=crucial point 1, cp2=crucial point 2

CRUCIAL POINTS THE PATH PASSES THROUGH	# OF PATHS FROM (0,0) TO CP1	# OF PATHS FROM CP1 TO CP2	# OF PATHS FROM CP2 TO (7,5)	# OF PATHS FROM CP1 TO (7,5)	# OF PATHS FROM (0,0) TO CP1 TO CP2 TO (7,5)
(0,0)**(0,4)(4,5)**(7,5)	$\binom{4}{4}=1$	$\binom{5}{1}=5$	$\binom{3}{0}=1$	5*1=5	1*(5+3)=8
(0,0)**(0,4)(5,4)**(7,5)		$\binom{5}{0}=1$	$\binom{3}{1}=3$	1*3=3	
(0,0)**(1,3)(4,5)**(7,5)	$\binom{4}{3}=4$	$\binom{5}{2}=10$	$\binom{3}{0}=1$	10*1=10	4*(10+15)=100
(0,0)**(1,3)(5,4)**(7,5)		$\binom{5}{1}=5$	$\binom{3}{1}=3$	5*3=15	
(0,0)**(4,1)(4,5)**(7,5)	$\binom{5}{1}=5$	$\binom{4}{4}=1$	$\binom{3}{0}=1$	1*1=1	5*(1+12+12+1)=130
(0,0)**(4,1)(5,4)**(7,5)		$\binom{4}{3}=4$	$\binom{3}{1}=3$	4*3=12	
(0,0)**(4,1)(6,2)**(7,5)		$\binom{3}{1}=3$	$\binom{4}{3}=4$	3*4=12	
(0,0)**(4,1)(7,1)**(7,5)		$\binom{3}{0}=1$	$\binom{4}{4}=1$	1*1=1	
(0,0)**(5,0)(5,4)**(7,5)	$\binom{5}{0}=1$	$\binom{4}{4}=1$	$\binom{3}{1}=3$	1*1*3=3	1*(3+12+3)=18
(0,0)**(5,0)(6,2)**(7,5)		$\binom{3}{2}=3$	$\binom{4}{3}=4$	1*3*4=12	
(0,0)**(5,0)(7,1)**(7,5)		$\binom{3}{1}=3$	$\binom{4}{4}=1$	1*3*1=3	

Fig. 7

Each path goes from (0,0) through one of the four first crucial points and then through a second crucial point to (7,5). For example, if the path went from (0,0) through a first crucial point (4,1), it could go one of four ways—through the second crucial point (4,5), (5,4), (6,2), or (7,1)—and finally to (7,5). Let p@(a,b)(m,n) denote the number of unique paths from (a,b) to (m,n). Then, the number of paths that go from (0,0) through (4,1) and ultimately to (7,5) is:

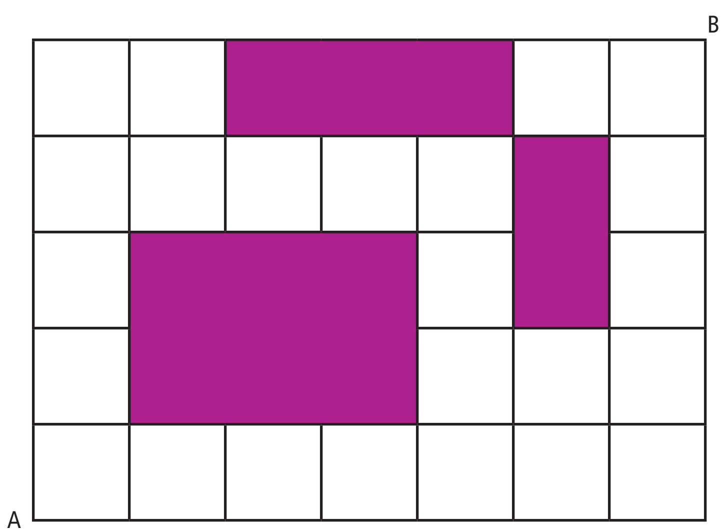


Fig. 8

k@(4,1)*[*p*@(4,1)(4,5)**p*@(4,5)(7,5)+*p*@(4,1)(5,4)**p*@(5,4)(7,5)+*p*@(4,1)(6,2)**p*@(6,2)(7,5)+*p*@(4,1)(7,1)**p*@(7,1)(7,5)]=130 (Fig. 7)

The same method, as shown above, can be used to calculate the paths that go through (0,4), (1,3), and (5,0). (Fig.7)

Adding the number of possible paths from (0,0) to (7,5) through each of the first crucial points, we get a total of 8+100+130+18=**256 paths.**

Further Reading: *The theory of lattice paths is an important branch of modern combinatorics; Brualdi gives a nice introduction in section 8.5.*

Bonus Problem: Given a 7x5 grid with 3 holes, how many different paths are there from A to B? (Fig. 8)

HIDDEN CIRCLES AND THE DIGITS OF Π

BY REINHARD ILLNER

The number π is defined as the ratio of a circular object's circumference to its diameter. In fith or sixth grade (I can't quite remember) the teacher asked us to take round objects (coffee cups, pots, balls, whatever we could find) and do this measurement. In preparing this essay, I repeated this age old exercise with my own coffee cup and a hygrometer (a small round device that measures the humidity in my office). Table 1 shows what I measured with a primitive tape measure. C and d denote measured perimeter and diameter and q, the computed quotient.

Object	C (cm)	d (cm)	q
Cup	27.7	8.8	3.14773
Hygrometer	26.4	8.4	3.14286

Table 1: Measurements

Since we all know that π=3.141592636..., it is clear that my method didn't provide great accuracy. Of course, I could have tried harder, with a better tape measure and larger round objects. I could have also repeated my measurement many times and computed averages; however, I do not wish to waste anybody's time. 2,200 years ago Archimedes proved π to be between 223/71 and 22/7, using regular polygons to approximate a circle (my high school teacher taught us that Archimedes had computed hundreds of digits and I believed this until a referee for this article taught me otherwise. According to Wikipedia, until the year 1,000, π was only known to less than ten digits [1],[2]). Now however, there are many algorithms and experiments one can use to approximate π, and we know more of π's fascinating properties, for example, that it is a transcendental number (meaning it is neither rational nor algebraic). This property follows from what is known as the Lindemann-Weierstrass Theorem. In particular, the digits of π will never repeat periodically, nor can we expect any pattern.

Let me mention a second simple experiment which can be used to approximate π. In elementary mechanics one studies the pendulum, and after some simplifications derives the relationship

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where T is the period and l is the length of the pendulum and $g = 9.80665\ m/sec^2$ is standard gravity (earth acceleration of a free falling object at sea level). The appearance of π in this formula may appear a bit mysterious until you understand that the solution of the differential equation governing the pendulum motion approximates a circular motion in phase space and circles, of course, are the underlying geometric objects. You could set the generic task: See π, find the circle.

It is a standard exercise in a physics lab to compute g by measuring T and l. Conversely, if one already knows g (i.e from free fall experiments), one can use the formula to approximate π. However, no higher level accuracy need be expected, because the formula itself is only an approximation (derived as a linear approximation from the true pendulum equations), reasonably accurate only for small amplitudes. In addition, measurement errors will affect the quantities T and l. If you wish to use this experiment to find π, your pendulum should be a string of length l, with an attached weight that is much heavier than the string (because the formula is derived for such conditions).

I did the experiment with a homemade pendulum (made of a string with a golf ball attached at one end). The length of this pendulum from pivot point to the center of the golf ball was 105 cm (I did the best I could measuring this) and this pendulum exhibited a frequency of 29 swings in 60 seconds. This gives T=2.068965... seconds. Inserting these data and the value of g into the formula, I found π≈3.1614....

I then lengthened the string, producing a pendulum with l=114.5 cm and measured that this pendulum swung 56 times in 120 seconds or T=2.142857... seconds. This gives π≈3.1346.... The average of both experiments is 3.148. Of course, it's off, but this 'accuracy' exceeded my expectations.

For practical purposes it should never be necessary to compute π to more than 50 digits, however, there is some interest in the methods themselves, as they tend to shed light on many truths (or hidden truths) in geometry and analysis.

Consider the well-known expansion, known as the Leibniz series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots.$$

A bit of playing with your pocket calculator will convince you that this is not a very good method to compute the digits of π: the first four terms on the right, multiplied by 4, give 2.895238…. One has to add many, many terms until reasonable accuracy is obtained (actually, you can read from formula how many terms you have to include to compute π to, for example, 100 digits. Give it a try). Nevertheless, the formula should intrigue you because as written, there is no hint why this should be true at all. The hidden truth is that this formula arises from an elementary trigonometric identity, an integration and a so-called power series expansion, all things we do in first year calculus, but which were at the forefront of mathematical research 250 years ago (Leonhard Euler, who would recently have celebrated his 306$^{\text{th}}$ birthday, did much of the research).

Here are the steps: We know that tan(π/4)=1: a line at 45 degrees counterclockwise from the horizontal has slope 1 (this is where the circle is hiding), so π/4 = $\tan^{-1}1$. From Calculus we know that

$$\tan^{-1}1 = \int_0^1 \frac{1}{1+x^2}\,dx,$$

and if we use the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots.$$

and integrate term by term, with $\int_0^1 x^k\,dx = \frac{1}{k+1}$, we find exactly Leibniz' formula.

Of course, we have to somehow justify the manipulations with infinitely many terms, but this can be done, and can be found in university textbooks on real analysis.

These steps show that

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \qquad (1)$$

which allows approximation of all values of the inverse tangent. Note that this works better (the series on the right converges faster) for smaller values of x.

Further, the trigonometric identity

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

can be used (with some work) to derive the formula (attributed to a man named Machin and first published in 1706)

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}. \qquad (2)$$

This offers a much better way to approximate π, because for x=1/5 and x=1/239 the right hand side of (1) converges much faster. Again, feel free to experiment with a pocket calculator, for example, the final displayed term in (1), evaluated for x=1/5, is 0.0000018286…. Much more on the power of Machin-like formulas to compute π with great accuracy can be found in [3, 4]. In combination with modern supercomputers, these formulas allow accuracies in the billions of digits.

A different and powerful formula is known as the Bailey-Borwein-Plouffe series

$$\pi = a_0 + \frac{1}{16}a_1 + \frac{1}{16^2}a_2 + \frac{1}{16^3}a_3 + \ldots.$$

where

$$a_n = \frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}$$

for n=0,1,2,.... Adding just the first two terms on the right gives 3.141422466.... This series converges to π at a very impressive rate.

Two other things that are rather remarkable are: first, the derivation of this identity requires no more than clever high school algebra and elementary integrations (involving trigonometric functions; this is, again, where the circle is hiding) and could have been done centuries ago (but was first published in only the 1990s [5]); second, in the hexadecimal system (based on the base 16 vs. 10 for the decimal system), this formula allows computing digits of π without knowing all previous digits.

In passing, the number π also shows up in the identities

$$e^{i\pi} + 1 = 0$$

(which nicely combines five fundamental real and complex numbers) and

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$$

(which is of major importance in statistics). I leave it as a challenge to the reader to find "where the circle is hiding" in these identities.

So, I have shown four ways of getting to π, two experimental and two computational. With more effort, time and information, one could fill a book with such methods. I will provide one more example, this time from a (hypothetical) billiard game. This was first pointed out by Gregory Galperin [6] and is a fairly recent observation.

Consider two billiard balls which move on a straight line (in one dimension), without gravitational forces or friction and with fully elastic collisions (meaning that collisions between the balls preserve momentum and kinetic energy). We present a scenario in which there is a solid elastic wall at x=0, ball A (with initial speed 0) has mass 1, radius 1 and is centered at x=3 and ball B has mass m≥1, radius 1, velocity −1 and is centered at x=6 (the radii and initial centers are of no actual importance; what matters is that the balls are initially apart and ball B is set up to hit ball A from the right at speed 1). See Figure 1.

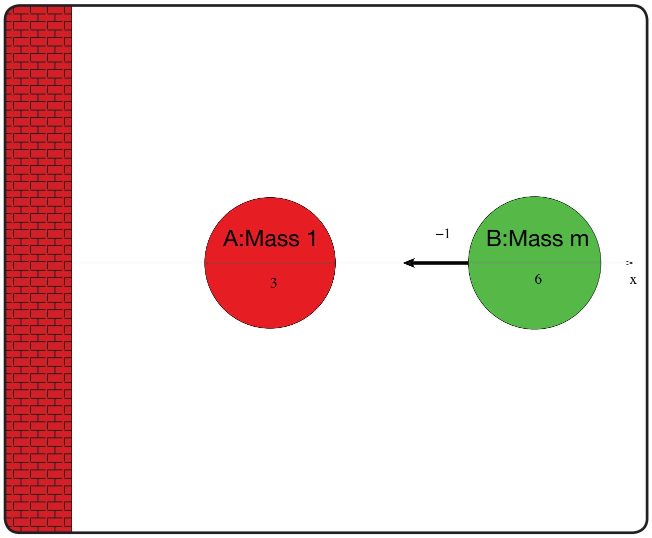


Figure 1: Two elastic spheres

If m=1, then the balls are of equal mass and the result is very predictable: ball B will hit ball A, they will exchange velocities, then ball A will hit the wall, bounce back with velocity +1, hit ball B a second time and ball B will fly off with velocity +1. We observe one wall collision and three collisions in all. What happens if ball B is heavier than ball A (m>1)? Momentum and energy conservation still uniquely determine the outcome of each collision: if, we let u_0 be the initial velocity of ball A (we took $u_0 = \mathbf{0}$) and the initial velocity of ball B (we took $v_0 = \mathbf{-1}$) then the velocities u'_0, v'_0 after the collision will be

$$u'_0 = u_0 - \frac{2m}{m+1}(u_0 - v_0) \qquad (3)$$

$$v'_0 = v_0 + \frac{2}{m+1}(u_0 - v_0) \qquad (4)$$

This is known as the collision transformation (a well-known concept in the kinetic theory of gases). One readily checks that

$$u'_0 + mv'_0 = u_0 + mv_0$$

(momentum conservation) and

$$(u'_0)^2 + m(v'_0)^2 = (u_0)^2 + m(v_0)^2 \qquad (5)$$

(energy conservation). The collision transformation is uniquely determined by these two properties.

Ball A will now bounce off the wall and head right; it will collide again with ball B, but as ball B is heavier than ball A, this may not be the last collision—ball A will head for the wall again, bounce back, meet again with ball B and so on. Figure 2 show what this looks like in *x,t* 'space' time; for convenience, the particles have been shrunk to points (we mentioned before that the size of the particles did not matter).

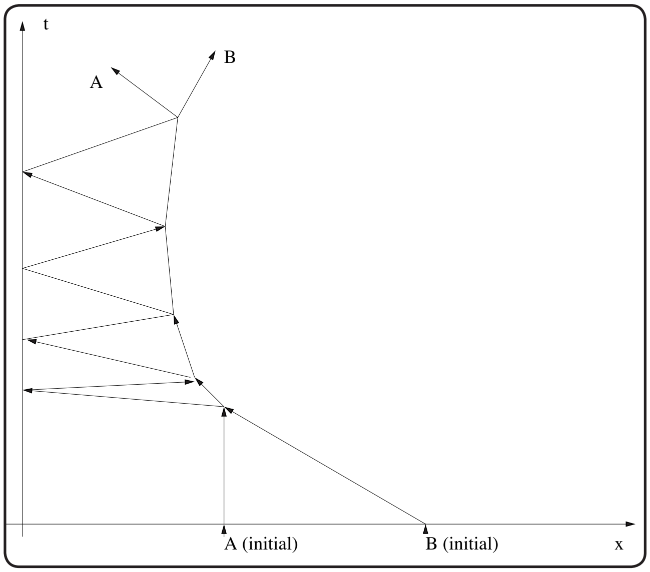


Figure 2: Many collisions in spacetime

We need some terminology to carry on. Suppose that $u_0, u_1, u_2, \ldots$ denote the velocities of sphere A initially, after the first wall bounce, then after the second wall bounce, etc. and that $v_0, v_1, v_2, \ldots$ denote the velocities of sphere B initially, after the first collision with A and then after the second collision with A, etc. From the collision transformation we find $u_1 = -u'_0$,
$v_1 = v'_0$, or

$$u_1 = \frac{m-1}{m+1}u_0 - \frac{2m}{m+1}v_0 \qquad (6)$$

$$v_1 = \frac{2}{m+1}u_0 + \frac{m-1}{m+1}v_0 \qquad (7)$$

The two particles were originally on collision course (or in a collision configuration) because $v_0 - u_0 = -1 < 0$ and if $v_1 - u_1 \leq 0$ they will collide again. We can then compute (u_2, v_2), (u_3, v_3) etc., until we find a number k such that, for the first time, $v_k - u_k > 0$. Particle A can then not catch up with particle B and there will be no more collisions.

Using a computer, one can find the number k with a little bit of effort.

Table 2 shows k as a function of m, the mass of particle B and, following Galperin's idea, we have taken $m = 100^n$ where n=0,1,2,3,….

m	N (total)	M (wall touches)
1	3	1
100	31	15
10,000	314	157
10^6	3142	…
10^8	31415	…

Table 2: Number of collisions

Here, N and M are the numbers of total collisions and wall collisions, respectively. Remember that particle A is initially at rest and particle B moves initially at $v_0 = -1$.

It appears that the number of all collisions (including the wall touches of ball A) produce the digits of π and the number of wall touches is half or one less than half of the corresponding approximation of π. The latter is easily understood- it is possible that balls A and B have a final collision such that A retains a positive velocity and will not return to the wall (as sketched in 2). But why should this experiment produce the digits of π at all? Where is the circle hiding? Before we answer this riddle, let us point out that this is really a thought experiment that can't be performed in reality; to get just four digits of π you have to have ball B a million times heavier than ball A and, of course, both have to be perfectly elastic and not be subject to friction or gravity.

But where is the circle? The explanation is hidden in the properties of the transformations (6, 7), although another idea is necessary. This idea has to do with the identity (5), which demonstrates the conservation of kinetic energy. It turns out that things become simpler if one rescales the speeds v_0, v_1, v_2 etc. of ball B by defining

$$w_0 = \sqrt{m}\,v_0, w_1 = \sqrt{m}\,v_1$$

etc.

The energy conservation (5) becomes then the simpler equation

$$(u'_0)^2 + (w'_0)^2 = (u_0)^2 + (w_0)^2 \qquad (8)$$

and the collision transformation (7) becomes

$$u_1 = \frac{m-1}{m+1}u_0 - \frac{2\sqrt{m}}{m+1}w_0 \qquad (9)$$

$$w_1 = \frac{2\sqrt{m}}{m+1}u_0 + \frac{m-1}{m+1}w_0 \qquad (10)$$

This is the same transformation as before, but the speed coordinate for ball B has been rescaled. In this new coordinate system, the equations (9, 10) are where the circle is hiding: If you set

$$\alpha = \frac{m-1}{m+1},\ \beta = \frac{2\sqrt{m}}{m+1}$$

then one immediately checks that $\alpha^2 + \beta^2 = 1$ and therefore there is an angle θ such that $\cos\theta = \alpha$, $\sin\theta = \beta$. Geometrically this means that in the u–w plane, (9,10) is a rotation in the counterclockwise sense by the angle θ and in our setup we begin the rotation with the initial point $(0, -\sqrt{m})$. The speeds (u_j, w_j) computed from repeated application of (9, 10) arise from repeated rotations by θ in the u–w plane for $j = 0,1,2,\ldots$ as shown in Figure 3 or as expressed by the transformation (rotation)

$$\begin{pmatrix} u_{j+1} \\ w_{j+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_j \\ w_j \end{pmatrix}$$

Energy conservation, as stated in (8), is the key ingredient in this: It implies that the collision transformation in this context must conserve the length of the vector (u_0, w_0) and only rotations or reflections do this.

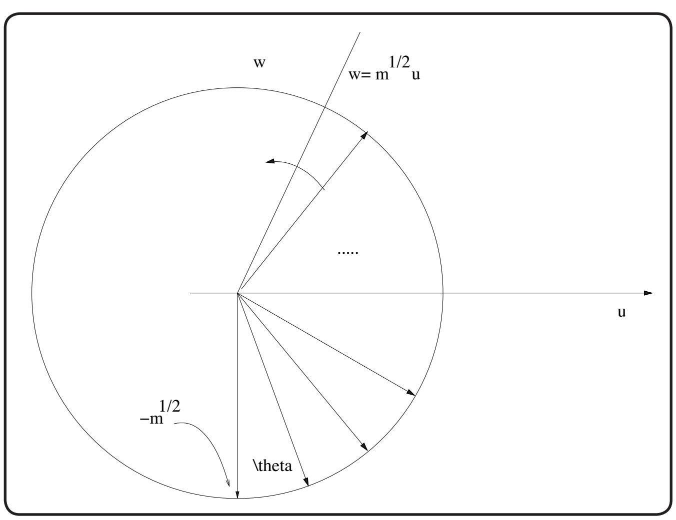


Figure 3: Collisions are rotations!

We are almost there! There will be no more collisions after the first k for which $v_k > u_k$ or, equivalently, $w_k > \sqrt{m}\,u_k$. Hence we have to find out for which k the sum of the angles will have crossed the line with slope $\sqrt{m}$. From the picture, this means we are looking for the smallest k for which $\tan(k\theta - \frac{\pi}{2}) > \sqrt{m}$.

Now, let us consider a large m. Then $\tan^{-1}\sqrt{m} \approx \frac{\pi}{2}$ (or, there have been enough collisions to go almost through a half-circle, meaning $k\theta \approx \pi$). We can also approximate θ in terms of m by observing that $\alpha = \cos\theta \approx 1 - \frac{\theta^2}{2}$, hence $\theta \approx \frac{2}{\sqrt{m+1}}$.

Putting it all together gives $k \approx \pi\frac{\sqrt{m+1}}{2}$ and this is an approximation of the expected number of wall touches: For example, for $m = 10^4$ we find $2k \approx 100\pi \approx 314$.

There is much more about this on the internet, in particular in article [6]. I would like to thank my colleague Peter Dukes for bringing this method of finding π to my attention.

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Complementary Events & the Pink Tail Problem

BY BILL RUSSELL, JAMES BOWIE HIGH SCHOOL, AUSTIN, TX

In a probability experiment, the complement of an event consists of all elements in the sample space not contained in the event. This concept is a basic, yet powerful, tool in probability. To illustrate this, consider the following situation:

At halftime at a basketball game, a fan can win a prize for making a basket from half court and is given five chances to do so. In reality, the shooter will stop taking shots if he makes one, but for this problem let's assume that he takes all five shots even if he makes one of the first four.

Suppose that the fan's shooting percentage from half court is 2%. We will further assume that his shots are independent of each other. That is, every time he takes a shot, his probability of making a basket is 0.02. We will disregard, for example, fatigue, which might make the probability decrease as he takes more shots.

Let's assign a random variable X to represent the number of shots made in five attempts. The six possible outcomes will be: $X = 0, X = 1, X = 2, X = 3, X = 4$ and $X = 5$. Most of these outcomes can occur in more than one way. For example, $X = 1$ if he makes the first shot, the second shot, the third shot, the fourth shot or the fifth shot.

The fan wins the prize if $X = 1, X = 2, X = 3, X = 4,$ or if $X = 5$. These events can also be described collectively by the inequality $X \geq 1$. In other words, he wins the prize if he makes at *least* one shot. Since these five events are pairwise *mutually exclusive* (that is, only one of them can possibly occur), the probability that one of them occurs can be found by adding the probabilities that each occurs. This would entail calculating five separate probabilities and adding them together. Although this is not an overwhelming task, there is a simpler way:

Of the six events described above, exactly one must occur. This means that

$$P(X = 0)+ P(X = 1)+\ldots+ P(X = 5)= P(X = 0)+ P(X \geq 1)=1. \qquad (1)$$

Therefore, rather than add the five probabilities described above, it is easier to figure the probability that $X = 0$ and subtract from 1.

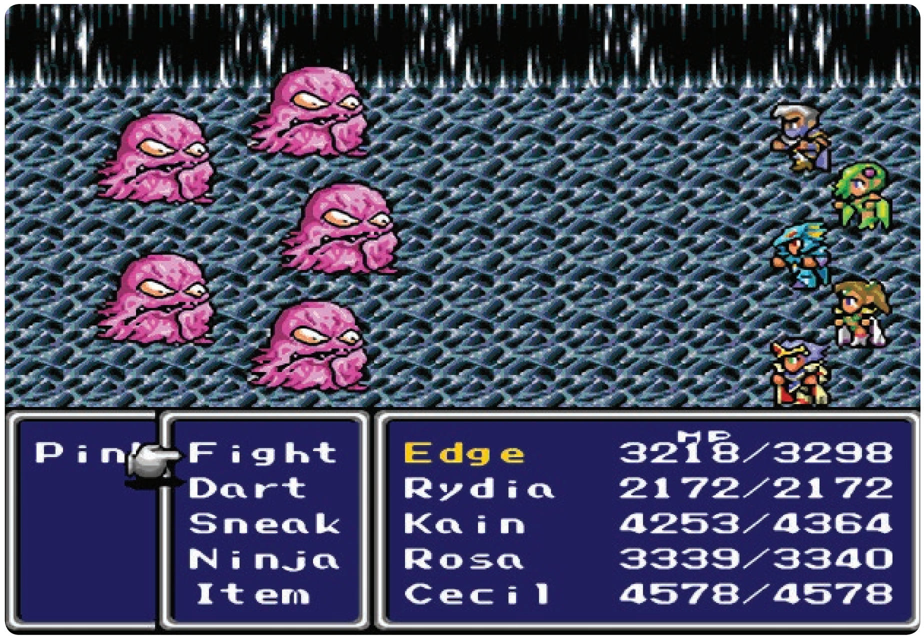
There is an additional benefit to using the complement of a compound event to calculate the probability of that event. By using complements, we can easily write a function that describes the probability of making at least one shot in terms of the number of shots taken. This generalizes the outcome of the problem and allows us to use the same process to answer multiple questions of the same type.

If the probability of making a shot is 0.02, then the probability of missing that shot is 0.98. If n independent shots attempts are made, then the probability that all will be missed is $(0.98)^n$. The complement of the event that the fan misses all the shots is the event that he makes at least one shot, so $P(X \geq 1)=1-0.98^n$.

The Pink Tail Problem

2011 marked the 20th anniversary of the US release of the landmark video game Final Fantasy II (FFII). FFII contained an item called the *Pink Tail*, which is one of the most rare and elusive items in video game history.

It is desirable because it can be exchanged for the *Adamant Armor* – the strongest armor in the game – and there is no other way to get this particular piece of equipment. Much misinformation and bad mathematics has been circulated about this item; we can use the notion of complements of events to correct these inaccuracies.



The only way to acquire a Pink Tail is to defeat a ferocious-looking enemy, *Pink Puff* (see picture) and have it drop one. Pink Puffs only appear in one room in the entire game and within that room, the probability that a random enemy group will consist of Pink Puffs is a mere $1/64$. Even if you find a group of Pink Puffs, the chances of getting this rare drop are miniscule: For each Pink Puff defeated, there is only a 5% chance that it will drop any item at all, and the probability that the item dropped will be a Pink Tail is only $1/64$. The only good news is that Pink Puffs appear in groups of 5 and these 5 enemies drop (or don't drop) items independently of each other.

The first problem, then, is finding Pink Puffs. From the above discussion, the probability that you will encounter of at least one group of Pink Puffs in n random encounters is

$$P(n) = 1 - \left(\frac{63}{64}\right)^n \qquad (2)$$

Here is a table of values for this function:

n	10	30	50	75	100
$P(n)$	0.15	0.38	.054	.069	.079

Table 1: Probabilities of finding at least one group of Pink Puffs in *n* random encounters.

Notice that even if you have 100 enemy encounters, there is still a 21% chance that you will not encounter any Pink Puffs. At 44 encounters, the chances are about 50%. Random encounters take roughly 20 seconds each, so after about 15 minutes the chances of encountering this enemy are about 50:50. The bottom line is that finding the enemies that drop Pink Tails can take considerable time.

Now let's look at the chances of actually getting Pink Puffs to drop the desired item. For the next four problems, assume that you have overcome the above odds and have now encountered a group of five Pink Puffs. Let's further assume that there is no chance that the Pink Puffs will win the battle, a reasonable assumption if your team is sufficiently powerful to have advanced to this stage of the game.

Problem 1: Find the probability that at least one enemy drops an item.

Recall that the probability that a Pink Puff drops any item is 0.05. Thus, P(At least one drop) = 1 – P(no drops) = 1 – $(0.95)^5 \approx 0.226$. So, at least one item will be dropped in slightly more than one encounter out of five. Looked at pessimistically, even when you are fortunate enough to get a Pink Puff encounter, you will get no dropped items 78% of the time.

Problem 2: Find the probability that a specific Pink Puff will NOT drop a Pink Tail.

There are two mutually exclusive ways that this can happen: either there is no drop, or there is a drop, but it is not a Pink Tail. So,

$$\text{P(no Pink Tail)} = \text{P(No drop)} + \text{P(getting a drop that is not a Pink Tail)} = 0.95 + (0.05) \times \frac{63}{64} \approx 0.99921875 \qquad (3)$$

Problem 3: Find the probability that none of the five Pink Puffs drop a Pink Tail.

Since the drops are independent, this probability is simply $(0.99921875)^5 \approx 0.9961$.

Problem 4: Find the probability of getting at least one Pink Tail.

From the above discussion, the probability of at least one Pink Tail in n Pink Puff battles is

$$T(n)=1-(0.9961)^n \tag{4}$$

Table 2 shows the values of this function for some values of n.

n	10	50	100	500	1000
$T(n)$	.038	.177	.323	.858	.980

Table 2: Probabilities of getting at least one Pink Tail from *n* encounters with Pink Puffs.

Remember that this table does not account for the low probability of even finding Pink Puffs! Now let's see how rare these drops actually are.

Problem 5: Find the probability that a random encounter will result in at least one Pink Tail.

Again,

P(At least one Pink Tail) = 1 – P(no Pink Tails) = 1 – [P(encounter is not with Pink Puffs) + P(encounter contains Pink Puffs but none drop a Pink Tail)]

$$= 1-\left[\frac{63}{64}+\frac{1}{64}(0.9961)\right] \approx (1-.99993606) \approx 6.1\times 10^{-5} \tag{5}$$

Thus, the probability of getting at least one Pink Tail after n random encounters is $1-(.99993906)^n$.

Table 3 shows some values for this function:

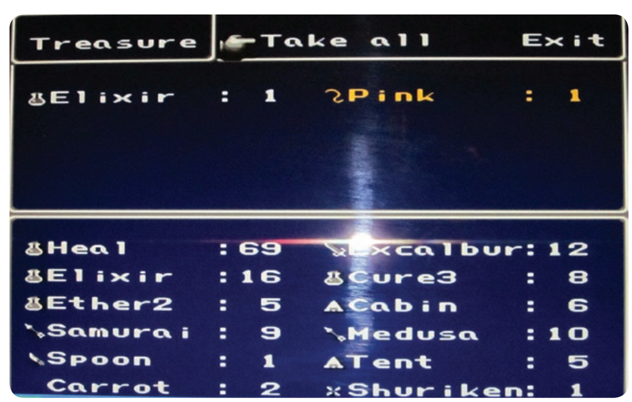
n	10	50	100	500	1000
$P(n)$	6 x 10^{-4}	.003	.006	.030	.059

Table 3: Probabilities of getting at least one Pink Tail from *n* random encounters.

Even with 1000 random encounters, there is less than a 6% chance of getting a Pink Tail if you go after this rare item in the traditional manner.

The probabilities in Table 3 are low because you are trying to get *three* low-probability events to all occur in the same encounter — finding Pink Puffs, getting them to drop an item and having that item be a Pink Tail. One way to circumvent this problem is to play a ROM version of the game so that you can save your game as soon as you encounter a group of Pink Puffs. That way, you only have to deal with the low probability of finding Pink Puffs once (as opposed to every time that you encounter enemies). Essentially, you can fight the same battle over and over again until you get a Pink Tail to drop, thus allowing you to deal with the more favorable probabilities in Table 2 rather than the ones in Table 3. In this scenario, your probability of getting a Pink Tail hits 50% when $n = \dfrac{\log 0.5}{\log 0.9961} \approx 177$, which seems a much more reasonable number than those illustrated in Table 3. However, since each battle takes about 1.5 minutes to complete, this still represents roughly 4.5 hours of fighting the same battle to get only a 50% chance of obtaining a single Pink Tail – and using the save-anywhere feature of a ROM is certainly less than ethical.

In light of all of this, it seems that the game designers went somewhat overboard in establishing the difficulty of obtaining this item. If an item is too difficult to obtain, then most gamers simply won't try. Perhaps the designers lacked the math skills to determine just how rare they made this particular item. In any event, finding a Pink Tail is clearly an extremely rare event. Consequently, the gamer who manages to acquire one of these rare items, can justifiably experience an exhilaration matched only by the thrill and excitement of having calculated the above probabilities!



COVERING SQUARE BOARDS WITH V-TROMINOES

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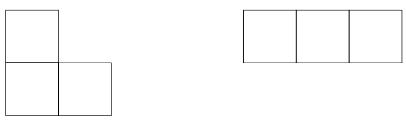
SECTION 1. INTRODUCTION.

A rectangular board is called a *punctured board* if one of its squares is removed. The diagram below shows the only two shapes, called *trominoes*, which are formed of three unit squares joined edge-to-edge. The one on the left is called the *V-tromino* and the one on the right is called the *I-tromino*. We are not concerned with the latter in this paper.



In his definitive treatise *Polyominoes*, Solomon W. Golomb proved that any punctured 2^k x 2^k board may be covered by copies of the V-tromino. By this, we mean that no copy protrudes beyond the board and no two copies overlap.

The argument is by mathematical induction. It is so instructive that we repeat it here. When k=0, the punctured 1×1 board is empty and may be covered by zero copies of the V-tromino vacuously. When k=1, a punctured 2×2 board is a copy of the V-tromino and may be covered by one copy of the V-tromino trivially.

Suppose the result holds for some $k \geq 1$. Consider a punctured $2^{k+1} \times 2^{k+1}$ board. Divide it into four equal quadrants. Without loss of generality, we may assume that the square removed is in the north-east quadrant. By the induction hypothesis, this punctured quadrant may be covered by copies of the V-tromino. We now place a copy of the V-tromino so that it covers the south-east corner of the north-west quadrant, the north-east corner of the south-west quadrant and the north-west corner of the south-east quadrant (see fig. 1). Then each of these quadrant is missing one square. By the induction hypothsis again, each of these punctured quadrants may be covered by copies of the V-tromino. By mathematical induction, the result holds for all $k \geq 0$.

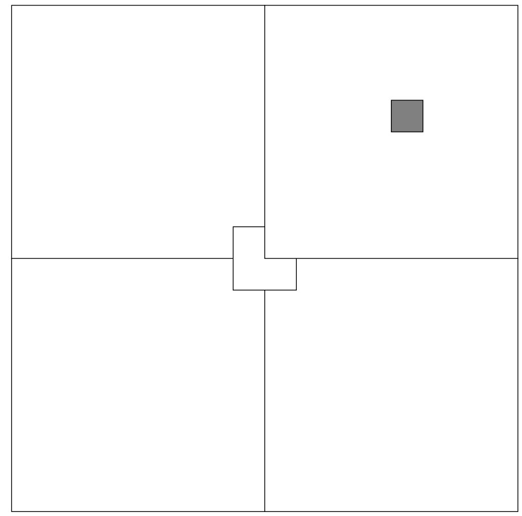
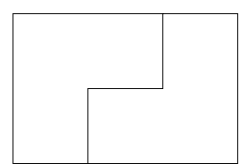


Fig. 1

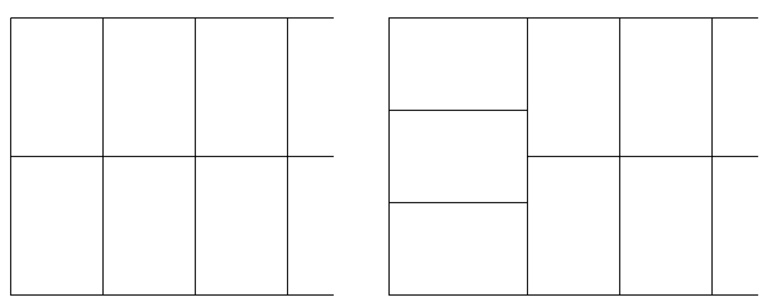
In this paper we consider $n \times n$ boards, punctured or otherwise, where n is not necessarily a power of 2.

SECTION 2. BASIC CONSTRUCTIONS.

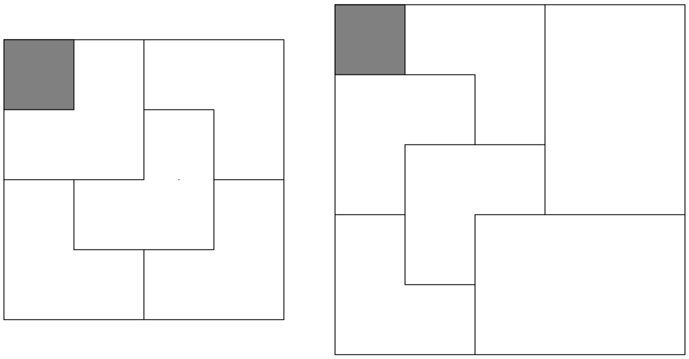
In this section, we give a few constructions which will be useful later. First, a 2×3 board may be covered with two copies of the V-tromino. From now on, if two copies of the V-tromino form a 2×3 subboard, we will just draw the subboard and not the individual copies.



The diagram below shows that a $6 \times n$ subboard may be covered by copies of the V-tromino for all $n \geq 2$. The case where n is even is on the left and the case where n is odd is on the right. From now on, $6 \times n$ subboards will be drawn without showing the individual copies of the V-tromino.

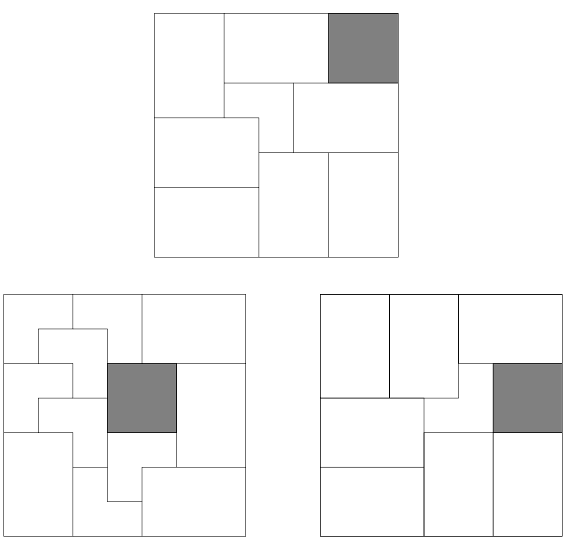


Finally, we will have occasions to make use of a 4×4 and a 5×5 subboard, each with a corner square removed. They may be covered by five and eight copies of the V-tromino respectively, as shown in the diagram below. The first subboard will be referred to as the *unusual* subboard and the second one the *special* subboard. Once again, they will be drawn without showing the individual copies of the V-tromino.



SECTION 3. COVERING OF PUNCTURED *nxn* BOARDS WHERE *n*≡1 (MOD 6).

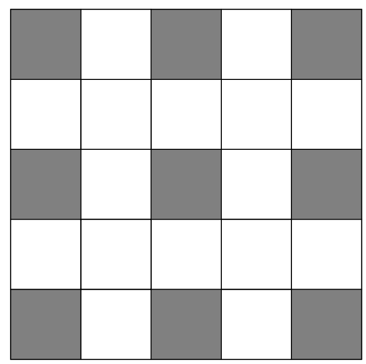
We prove by induction on k that all punctured $(6k+1)\times(6k+1)$ boards may be covered by copies of the V-tromino. The case $k=0$ is vacuously true and in any case covered by Golomb's classic result. For $k=1$, consider any punctured 7×7 board. The diagram below shows that by symmetry, the square removed must come from one of the shaded 2×2 subboards. The remaining part of the board may be covered by V-trominoes. Since a punctured 2×2 board may also be covered by Golomb's classic result, so may any punctured 7×7 square.



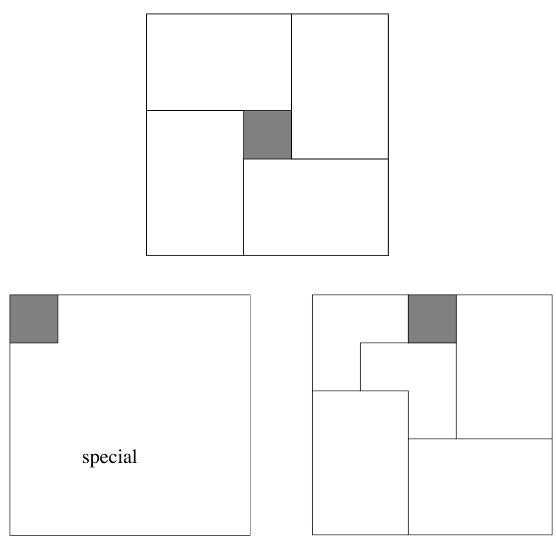
Consider now a punctured $(6k+1)\times(6k+1)$ board for some $k\geq 2$. By symmetry, we may assume that the square removed is in the $(6(k-1)+1)\times(6(k-1)+1)$ subboard at the north-east corner. Then the rest of the board may be divided into a $6\times(6k+1)$ subboard and a $6\times(6(k-1)+1)$ subboard. By the induction hypothesis and the results of Section 2, any punctured $(6k+1)\times(6k+1)$ board may be covered by copies of the V-tromino.

SECTION 4. COVERING OF PUNCTURED *nxn* BOARDS WHERE *n*≡ 5 (MOD 6).

It is not true that an arbitrary punctured 5×5 board may be covered by copies of the V-tromino. Consider the nine shaded squares in the diagram below. A copy of the V-tromino can cover at most one of them. Now we need exactly eight copies of the V-tromino to cover a punctured 5×5 board. Hence the square removed must be one of the nine shaded squares.



On the other hand, the diagram below shows that the square removed can be any of the nine shaded squares.



We claim that any punctured $(6k+5)\times(6k+5)$ board may be covered by copies of the V-tromino for any $k\geq1$. Such a board may be divided into a $(6k+1)\times(6k+1)$ suboard, two $6k\times4$ subboard and a special subboard. By symmetry, we may assume that the squared removed is in the $(6k+1)\times(6k+1)$ subboard. Our claim follows from the results in Sections 2 and 3.

SECTION 5. COVERING OF PUNCTURED *nxn* BOARDS WHERE N≡4 (MOD 6).

The case of a punctured 4×4 board is covered by Golomb's classic result. We claim that any punctured $(6k+4)\times(6k+4)$ board may be covered by copies of the V-tromino for any $k\geq1$. Such a board may be divided into a $(6k+1)\times(6k+1)$ suboard, two $6k\times3$ subboard and an unusual subboard. By symmetry, we may assume that the squared removed is in the $(6k+1)\times(6k+1)$ subboard. Our claim follows from the results in Sections 2 and 3.

SECTION 6. COVERING OF PUNCTURED *nxn* BOARDS WHERE N≡2 (MOD 6).

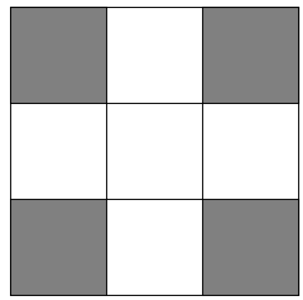
The case of a punctured 2×2 board is covered by Golomb's classic result. We claim that any punctured $(6k+2)\times(6k+2)$ board may be covered by copies of the V-tromino for any $k\geq1$. If $k=2h$, then the board may be divided into four $(6h+1)\times(6h+1)$ quadrants. If $k=2h+1$, then the board may be divided into four $(6h+4)\times(6h+4)$ quadrants. By symmetry, we may assume that the square removed is in the north-east quadrant. We now place a copy of the V-tromino so that it covers the south-east corner of the north-west quadrant, the north-east corner of the south-west quadrant and the north-west corner of the south-east quadrant, just as in Golomb's classic result. Then each of these quadrant is missing one square. The claim is justified by the result in Section 3 in the case $k=2h$ and by the the result in Section 5 in the case $k=2h+1$.

SECTION 7. COVERING OF *nxn* BOARDS WHERE N≡0 (MOD 3).

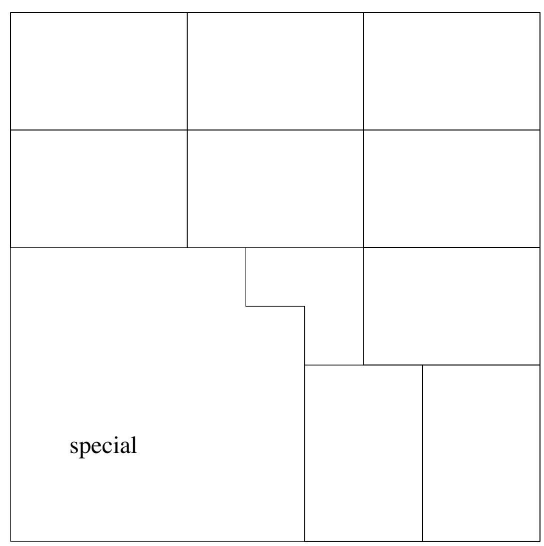
The total number of squares in a $3k\times3k$ board is $9k^2$ and no square will be removed. For $k=2h$, a $6h\times6h$ board may be divided into h copies of $6\times6h$ subboards.

By the results in Section 2, this board may be covered by copies of the V-tromino. Henceforth, we let $k=2h+1$.

The 3×3 board cannot be covered by copies of the V-tromino. Consider the four shaded squares in the diagram below. A copy of the V-tromino can cover at most one of them. Now we need exactly three copies of the V-tromino to cover a 3×3 board. Hence one of the four shaded squares will not be covered.



We prove by induction on h that a $(6h+3)\times(6h+3)$ boards may be covered by copies of the V-tromino for any $h\geq1$. The 9×9 board may be covered by 27 copies of the V-tromino as shown in the diagram below.



Consider now a $(6h+3)\times(6h+3)$ board for some $h\geq2$. It may be divided into a $(6(h-1)+3)\times(6(h-1)+3)$ subboard, a $6\times(6h+3)$ subboard and a $6\times(6(h-1)+3)$ subboard. By the induction hypothesis and the results in Section 2, any $(6h+3)\times(6h+3)$ board with $h\geq1$ may be covered by copies of the V-tromino.

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